M.A./M.Sc. (Mathematics) (2005 Pattern) Examination, 2010
MT 706 : NUMERICAL ANALYSIS (Old)

Time : 3 Hours Max. Marks : 80

N.B.: *i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*
iii) Use of non-programmable scientific calculators is ***allowed****.*

1. a) Determine the order of approximation for the sum and product of the expansions;

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + o(h^4) \text{ and}$$

$$\cosh = 1 - h^2/2! + \frac{h^4}{4!} + o(h^6).$$ **8**

b) Investigate the nature of the iteration $p_{n+1} = g(p_n)$ for the function

$$g(x) = 1 + x - \frac{x^2}{4}.$$ **8**

2. a) Perform four iterations of bisection method to solve x sin x = 1 on [0, 2]. **8**

b) Suppose Newton-Raphson iteration produces a sequence $\{p_n\}_{n=0}^{\infty}$ that converges to the multiple root P of order M of f(x). Then prove that the convergence is linear. **8**

3. a) For the linear system

$x^2 - y - 0.2 = 0$

$y^2 - x - 0.3 = 0,$

start with $(p_0, q_0) = (1.2, 1.2)$ and use Newton's method to compute (p_1, q_1) and (p_2, q_2). **8**

P.T.O.

b) Find the triangular factorization $A = LU$ for the matrix. **8**

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix}$$

4\. a) Solve the following system by Gauss-Seidel method.

$$\left.\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15 \end{aligned}\right\}$$ start with (1, 2, 2) and perform two iterations. **8**

b) Prove that the Jacobi iterations converge to the solution of the linear system $Ax = b$ starting with any initial vector $x^{(0)}$ provided that the matrix A is strictly diagonally dominant. **8**

5\. a) Let $f(x) = {8x}/{2^x}$. Use cubic Lagrange interpolation based on the nodes $x = 0, 1, 2, 3$, to approximate f(7.5). Compare with true value. **8**

b) Construct a divided difference table for $f(x) = \cos x$ based on the five nodes $x = 0, 1, 2, 3, 4$. Hence find $P_2(1.5)$. **8**

6\. a) Use Taylor expansions and derive the central-difference formula :

$$f'(x) = (f(x+h) - f(x-h)) / 12h .$$ **8**

b) Use the numerical differentiation formula

$f''(x_0) = [-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}] / 12h^2$, and $h = 0.1$ to approximate $f''(1)$ for the function $f(x) = x^6$. Compare with true value. **8**

7. a) Derive Trapezoidal rule for numerical integration and hence find the value of π by evaluating $\int_0^1 \frac{1}{1+x^2}\, dx$. **8**

b) Determine the degree of precision of the Simpson's $3/8$ rule. **8**

8. a) Use Runge-Kutta method RK4 and compute the numerical solution of the system

$$\left.\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 3x + 2y \end{aligned}\right. \quad \text{with} \quad \left\{\begin{aligned} x(0) &= 6 \\ y(0) &= 4, \end{aligned}\right.$$

at t = 0.02. **8**

b) For any fixed θ, show that

$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix. **4**

c) Construct Householder matrix P for w = $[0, 0, 1]^T$. **4**

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[3721] – 47

M.A./M.Sc. Examination, 2010
MATHEMATICS
MT 807 : Combinatorics (Old)
(2005 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Attempt **any five** questions.*

*2) Figures to the **right** indicate **full** marks.*

1. A) What is the number of ways that a five card hand has :

   i) each of the four values Ace, King, Queen and Jack ?

   ii) the same number of hearts and spades ? **6**

   B) How many arrangements of $5\ \alpha' s$, $5\beta' s$ and $5\gamma' s$ are there with atleast one β and atleast one γ between each successive pair of α's ? **6**

   C) Prove the following binomial identity using combinatorial argument

$$\binom{n}{0}+\binom{n+1}{1}+\ldots+\binom{n+r-1}{r-1}+\binom{n+r}{r}=\binom{n+r+1}{r}.$$ **4**

2. A) If there are n-objects with r_1 of type 1, r_2 of type 2, ..., r_m of type m, where $r_1 + r_2 + \ldots + r_{m-1} + r_m = n$, then the number of arrangements of these n objects denoted by P (n; r_1, r_2, ... r_m). Prove by mathematical induction that

$$P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\cdot\binom{n-r_1}{r_2}\cdot\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1 \ldots\ldots r_{m-1}}{r_m} = \frac{n!}{r_1!\, r_2! \ldots r_m!}.$$ **6**

   B) How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$? How many solutions with $x_i \geq 1$? How many solutions with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$? **6**

   C) Find the number of ways to get 25 rupees from 10 distinct people, if a person can give either 3 rupees, 8 rupees or none, using generating function. **4**

P.T.O.

3. A) Explain why $(1 + x + x^2 + x^3 + x^4)^r$ is not a proper generating function for the number of ways to distribute r-jelly beans among r-children with no child getting more than four jelly beans. **6**

   B) Show with generating functions that every positive integer can be written as a unique sum of distinct powers of 2. **6**

   C) Show that the number of partitions of an integer r as a sum of m positive integers is equal to the number of partitions of r, as a sum of positive integers, the largest of which is m. **4**

4. A) Using exponential generating function find how many r-digit quaternary sequences are there in which the total number of 0's and 1's is even ? **6**

   B) Build a generating function using summation method for $a_r = (r + 1)r(r - 1)$. **6**

   C) Find the number of 7-bead necklaces distinct under rotations using three black and four white beads. **4**

5. A) State and prove Burnside's Theorem. **8**

   B) Suppose we draw n-straight lines on a piece of paper so that every pair of lines intersect (but no three lines intersect at a common point). Use recurrence relation and find into how many regions do these n lines divide the plane. **8**

6. A) State and prove the Inclusion-Exclusion Formula. **6**

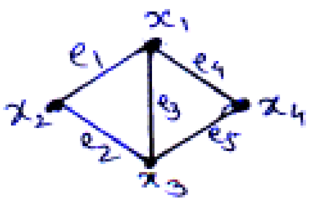
   B) How many different 3-colorings of the bands of an n band baton are there if baton is unoriented ? **6**

   C) Solve the following recurrence relation

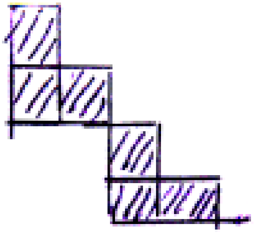
   $a_n = a_{n-1} + 3(n - 1)$, $a_0 = 1$. **4**

7. A) Using Inclusion-Exclusion theorem, find the number of n digit ternary sequences with atleast one 0, atleast one 1 and atleast one 2. **6**

B) How many ways are there to color the four vertices in the graph shown below with n colors such that vertices with a common edge must be different colors ? **6**



C) Find the rook polynomial for the following figure. **4**



8. A) Find the pattern inventory of black-white edge colorings of a tetrahedron. **6**

B) How many arrangements of the letters a, e, i, o, u, x, x, x, x, x, x, x, x (8 x′s) are there if no two vowels can be consecutive ? **6**

C) Find the number of different r-arrangements of objects chosen from unlimited supplies on n types of objects, using exponential generating function. **4**

[3721] – 101

M.A./M.Sc. (Semester – I) Examination, 2010
MATHEMATICS (2008 Pattern)
MT : 501 : Real Analysis – I

Time: 3 Hours Max. Marks: 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) ***All*** *questions carry* ***equal*** *marks.*

1. a) State and prove Cauchy-Schwarz's inequality. **6**

b) Show that the set of rational numbers is countable. **5**

c) Suppose A is any set and P(A) is its power set. Is any map $F : A \to P(A)$ onto ? Justify. **5**

2. a) Show that $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$ defines a metric on $(0, \infty)$. **6**

b) Give an example of a sequence $\{f_k\}_{k=1}^{\infty}$ of non-negative measurable functions on A, where $A \in M$ and $f = \lim\limits_{k\to\infty} \inf f_k$ on A such that $\int_A f \, dm < \lim\limits_{k\to\infty} \inf \int_A f_k \, dm$. **5**

c) Show that compact subsets of a metric space are closed. **5**

3. a) Let $A \subset (M, d)$ then prove that $x \in \overline{A}$ iff $B_{\in}(x) \cap A \neq \phi$ for every $\in > 0$. **6**

b) Is Cantor set compact ? What is its interior ? Explain. **6**

c) With usual notations, show that, $L^p(\mu)$ is a linear space where $1 \leq p < \infty$. **4**

4. a) Define a measurable function on $\mathbb{R}^n$ and show that following statements are equivalent. **8**

i) $\{x / f(x) > a\}$ is measurable for every $a \in \mathbb{R}$

ii) $\{x / f(x) \geq a\}$ is measurable for every $a \in \mathbb{R}$

iii) $\{x / f(x) < a\}$ is measurable for every $a \in \mathbb{R}$

iv) $\{x / f(x) \leq a\}$ is measurable for every $a \in \mathbb{R}$

b) Find limit points of $\mathbb{Q}$ and $\left\{\frac{1}{n}\right\}$ where $n \in \mathbb{N}$. **4**

c) Show that $\mathbb{R}$ with discrete metric space is not separable. **4**

5. a) State and prove Monotone Convergence Theorem. **5**

b) Draw the following graphs in $\mathbb{R}^2$. **6**

i) $\{\|u\|_1 < 1\}$ ii) $\{\|u\|_2 < 1\}$ iii) $\{\|u\|_\infty < 1\}$ for $u \in \mathbb{R}^2$.

c) Show that $\sigma : [0,1] \to [a,\ b]$ defined by $\sigma(t) = a + t(b-a)$ is homeomorphism and $f \in C[a, b]$ if $f_0\sigma \in C[0, 1]$. **5**

6. a) State and prove Holder's inequality. **6**

b) Show that a Riemann integrable function is also a Lebesgue integrable. **5**

c) Suppose $\{F_n\}$ is a decreasing sequence of non-empty closed sets in a complete space (M, d), with diam $F_n \to 0$, as $n \to \infty$ then show that $\bigcap_{n=1}^{\infty} F_n \neq \phi$. **5**

7. a) Show that $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}\right\}$ where $n, m \in \mathbb{N}$ forms an orthonormal set in $L^2([-\pi, \pi])$. **6**

b) Show that (M, d) is compact then every open cover of M has a finite subcover. **8**

c) State Banach contraction principle. **2**

8. a) Show that any non-empty complete metric space is of second category. **6**

b) State and prove Arzela-Ascoli Theorem. **8**

c) Show that evey continuous function is measurable. **2**

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[3721] – 102

M.A./M.Sc. (Semester – I) Examination, 2010
MATHEMATICS (2008 Pattern)
MT-502 : Advanced Calculus

Time : 3 Hours　　　　Max. Marks : 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*
3) Notation : $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ denote standard basis for $\mathbb{R}^n$.

1. a) Assume that $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable scalar field at a point $\vec{a}$ in Int S with total derivative $T_{\vec{a}}$; Then prove that $f'(\vec{a};\vec{y})$ exists for every $\vec{y} \in \mathbb{R}^n$. **6**

b) Let $\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector field, and let $\vec{f}(\vec{x}) = f_1(\vec{x})\vec{e}_1 + ... + f_m(\vec{x})\vec{e}_m$, where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, i = 1, 2, ..., m are scalar fields. Then prove that $\vec{f}$ is continuous if and only if component function f_i is continuous. **4**

c) Let $\vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\vec{g} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be two vector fields defined as :

$\vec{f}(x, y) = e^{x+2y}\vec{e}_1 + \sin(y + 2x)\vec{e}_2$ and

$\vec{g}(u, v, w) = (u + 2v^2 + 3w^3)\vec{e}_1 + (2v - u^2)\vec{e}_2$.

Compute $D\vec{h}(1,-1,1)$, where $\vec{h} = \vec{f}o\vec{g}$. **6**

2. a) If $\vec{f}$ is a vector field, show that $\vec{f}$ is differentiable at $\vec{a}$ then it is continuous at $\vec{a}$. **4**

b) Find the directional derivative of the scalar field f(x, y) = $x^2 - 3xy$ along the parabola $y^2 = x^2 - x + 2$ at the point (1, 2). **4**

c) State and prove chain rule for derivatives of vector fields. **8**

P.T.O.

3. a) Define line integral of a vector field along the curve. Illustrate by an example that line integral is independent of the path along a curve joining the two points. **8**

b) Give an example of a vector field $\vec{f}(x, y)$ defined on a open set $S \subset \mathbb{R}^n$ such that $D_1\vec{f}_2 = D_2\vec{f}_1$ but $\vec{f}$ is not gradient on S. **8**

4. a) Prove that the line integral of a continuous gradient is zero around every piecewise smooth closed path in an open connected set S in $\mathbb{R}^n$. **8**

b) i) Evaluate $\int_C \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$, where C is the circle $x^2 + y^2 = 4$ traversed in a counter clockwise direction.

ii) Let $\vec{f}(x, y) = \frac{-y}{x^2+y^2}\vec{e}_1 + \frac{x}{x^2+y^2}\vec{e}_2$ for $(x, y) \neq (0, 0)$. Show that $\int_C \vec{f}(x,y)$ is not zero, where C is the circle of radius $a > 0$ with center at origin. **8**

5. a) Prove that a continuous function f on a rectangle Q is integrable on Q. **8**

b) Evaluate $\iint_Q xy(x+y)dxdy$, where $Q = [0, 1] \times [0, 1]$. **4**

c) Evaluate $\iint_Q \sin^2 x \sin^2 y \, dxdy$, where $Q = [0, \pi] \times [0, \pi]$. **4**

6. a) State Green's theorem for plane region and verify it by an example. **8**

b) Evaluate $\iiint_S xyzdxdydz$, where

$S = \{(x,y,z) / x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$. **4**

c) State only the general formula for change of variables in double integrals and explain the terms involved. **4**

7. a) Define surface integral and explain the terms invovled in it. **6**

b) Let $x^2 + y^2 + z^2 = 1$ be a sphere of radius one. Find the fundamental vector product in explicit form of this sphere. Also discuss the singular points of this surface. **6**

c) If $\vec{r}(u,v) = (x_0 + a_1u + b_1v)\vec{e}_1 + (y_0 + a_2u + b_2v)\vec{e}_2 + (z_0 + a_3u + b_3v)\vec{e}_3$,

find $\frac{\partial \vec{v}}{\partial u} + \frac{\partial \vec{v}}{\partial v}$ in terms of u and v. **4**

8. a) State and prove divergence theorem. **6**

b) Show that curl $(\text{grad}\,\phi) = 0$. **4**

c) Use transformation formula to transform the integral $\iiint\limits_S f(x, y, z)\,dxdydz$, where S is sphere of radius a by using $x = \rho\cos\theta\cos\phi$, $y = \rho\sin\theta.\cos\phi, z = \rho\sin\phi$. **6**

M.A./M.Sc. (Semester – I) Examination, 2010
MATHEMATICS (2008 Pattern)
MT 503 : Linear Algebra

Time : 3 Hours Max. Marks : 80

Instructions *: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Let V be a finite dimensional vector space over K, and let X and Y be finite subsets of V. If Y is linearly independent and $V = \langle X \rangle$, prove that $|Y| \leq |X|$ **6**

b) Let V and V' be finite dimensional vector spaces over K. Prove that $V \simeq V'$ if only if dim V = dim V'. **6**

c) If X and Y are subspaces of a vector space V such that V/X and V/Y and finite dimensional, prove that the quotient space $V/(X \cap Y)$ is also finite dimensional. **4**

2. a) Let V_1, ...V_m be vector spaces over a field K. Prove that $V = V_1 \oplus \ldots \oplus V_m$ is finite dimensional if and only if each V_i is finite dimensional. **6**

b) Let D be the differential operator on $\mathbb{R}_3[x]$, write the matrix representation of D with respect to the ordered basis { $1 + x$, $x + x^2$, x^2+x^3, $x+ x^3$}. **6**

c) Prove that the geometric multiplicity of an eigenvalue of a linear operator cannot exceed its algebraic multiplicity. **4**

3. a) Let B be an ordered basis of an n-dimensional vector space V over K. If S and T are linear operators on V, Prove that $[S_0\ T]_B = [S]_B\ [T]_B$ and T is a bijection if and only if $[T]_B$ is an invertible matrix. **6**

b) Let V be a finite dimensional vector space over K and let T be a linear operator on V. If X and Y are T- invariant subspaces of V and $V = X \oplus Y$, prove that $X°$ and $Y°$ are $T°$-invariant subspaces of $V°$ and $V° = X° \oplus Y°$. **6**

c) Let K be a field and let $p(x) = x^n + a_{n-1} x^{n-1} + ... + a_0$ be a monic polynomial of degree n. Let A be an n×n matrix given by : **4**

$$A = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \cdots & 0 & -a_1 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

Prove that the characteristic polynomial of A is p(x).

4. a) Let V be a finite dimensional vector space over K of dimension n and let T be a linear operator on V. If $m_T(x) = p(x)^r$, where p (x) is a monic irreducible polynomial of degree m, prove that m divides n. **5**

b) Prove that two diagonalizable linear operators S and T on V are simultaneously diagonalizable if and only if they commute, that is ST = TS. **7**

c) Prove that a Jordan chain consists of linearly independent vectors. **4**

5. a) Let V be a finite dimensional inner product space and let f be a linear functional on V. Prove that there exists a unique vector x in V such that f (v) = (v, x), for all v in V. **7**

b) Let V and W be finite dimensional inner product spaces and let T ∈ < (V, W). Prove that there exists a unique linear mapping $T^* : W \to V$ such that for all v∈ V and w∈ W, $(Tv, w) = (v, T^*w)$. **5**

c) Prove that a Jordan subspace for a linear operator T is T-cyclic. **4**

6. a) Prove that a self adjoint operator T on a finite dimensional inner product space V is orthogonally diagonalizable. **6**

b) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$ find a polar decomposition of A. **6**

c) Let T be a unitary operator on V, dim V = n. If B_1 and B_2 are ordered orthonormal basis of V, prove that ${}_{B_2}[T]_{B_1}$ is a unitary matrix. **4**

7. a) Prove that a bilinear form is reflexive if and only if it is either symmetric or alternating. **6**

   b) Let A, B $\in$ $K^{n \times n}$. Prove that bilinear spaces $\left(k^n, \theta_A\right)$ and $\left(k^n, \theta_B\right)$ are isomorphic if and only if A and B are congruent matrices. **6**

   c) Let ϕ be a nondegenerate reflexive bilinear form on a finite dimensional vector space V over K. For a subspace S of V, prove that $S^{\perp\perp} = S$. **4**

8. a) Prove that a symmetric bilinear form on a finite dimensional vector space V over a field K of characteristic not equal to 2 is diagonalizable. **6**

   b) Prove that two triangulable n$\times$n matrices are similar if and only if they have the same Jordan canonical form. **6**

   c) Give all possible Jordan canonical forms if the characteristic polynomial is $(x – 2)^3 (x – 5)^2$. **4**

[3721] – 105

M.A./M.Sc. (Semester – I) Examination, 2010
MATHEMATICS
MT – 505 : Ordinary Differential Equations
(2008 Pattern)

Time : 3 Hours   Max. Marks : 80

N.B.: 1) Answer ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Find the general solution of $y'' - y' - 2y = 4x^2$. **5**

   b) If $q(x) < 0$, and if u(x) is a nontrivial solution of $u'' + q(x)u = 0$, prove that u(x) has at most one zero. **5**

   c) Let y(x) and z(x) be nontrival solutions of $y'' + q(x)y = 0$ and $z'' + r(x)z = 0$, where q(x) and r(x) are positive functions such that q(x) > r(x). Prove that y(x) vanishes at least once between any two successive zeros of z(x). **6**

2. a) Find the general solution of $(1 + x^2)\, y'' + 2xy' - 2y = 0$ in terms of power series in x. **5**

   b) Verify that the origin is a regular singular point and calculate two independent Frobenius series solutions for the equation $4xy'' + 2y' + y = 0$. **8**

   c) Are the functions $\phi_1(x) = \sin x$ and $\phi_2(x) = e^{ix}$ defined on $-\infty < x < \infty$ linearly independent ? Why ? **3**

3. a) Find the general solution of $(2x^2 + 2x)\, y'' + (1 + 5x)\, y' + y = 0$ near the singular point x = 0. **8**

   b) Find the general solution of the system **8**

$$\frac{dx}{dt} = 3x - 4y$$

$$\frac{dy}{dt} = x - y$$

P.T.O.

4. a) If a is an arbitrary constant, prove that the system $\frac{dx}{dt} = ax - y$, $\frac{dy}{dt} = x + ay$ has the origin as only its critical point, find the differential equation of the paths and solve this equation to find the paths. **6**

b) If $a_1 b_2 - a_2 b_1 \neq 0$, show that the system $\frac{dx}{dt} = a_1 x + b_1 y, \frac{dy}{dt} = a_2 x + b_2 y$ has infinitely many critical points, none of which are isolated. **6**

c) Show that $y(x) = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval, and find the particular solution for which $y(0) = 2$ and $y'(0) = 3$. **4**

5. a) Solve the following initial value problem by Picard's method and compare the result with exact solution **8**

$$\frac{dy}{dx} = 2x(1+y), y(0) = 0.$$

b) Show that the function $f(x,y) = xy^2$ satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq y \leq d$ but it does not satisfy a Lipschitz condition on any strip $a \leq x \leq b$ and $-\infty < y < \infty$. **8**

6. a) Let x_0 be an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$, and let a_0, a_1 be arbitrary constants. Prove that there exists a unique function y(x) that is analytic at x_0, is a solution of above differential equation in a certain neighbourhood of this point, and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. **8**

b) Find the eigenvalues and eigenfunctions of **8**

$$y'' - 4\lambda y' + 4\lambda^2 y = 0 \; ; \; y'(1) = 0, \; y(2) + 2y'(2) = 0.$$

7. a) Find a recurrence formula and the indicial equation for an infinite series solution around x = 0 for the differential equation $8x^2y'' + 10xy' + (x-1)y = 0$. **8**

b) Solve $y^{(4)} = 5x$ by variation of parameters. **8**

8. a) Find the general solution near x = 0 of the hypergeometric equation

$x(1-x)\,y'' + [c-(a+b+1)x]y' - aby = 0$ where a, b, and c are constants. **8**

b) Let ϕ be any solution of $L(y) = y'' + a_1 y + a_2 y = 0$, on an interval I containing a point x_0. Prove that for all x in I $\|\phi(x_0)\|\, e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\|\, e^{k|x-x_0|}$.

where $\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{1/2}$, $k = 1 + |a_1| + |a_2|$.

8

M.A./M.Sc. (Sem. – II) (2008 Pattern) Examination, 2010
MATHEMATICS
MT 601 : General Topology (New)

Time : 3 Hours Max. Marks : 80

***N.B.**: i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate marks.*

1. A) Define a basis for a topology on a set X. Show that the countable collection B = {(a, b / a < b, a and b are rational } is a basis that generates the standard topology on $\mathbb{R}$. **6**

   B) Show that the intersection of two topologies on a set X is a topology on X. Show that union of two topologies on X need not be a topology. **5**

   C) Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projection maps. Prove that π_1 and π_2 are open maps. Further, prove that the collection $S = \{\pi_1^{-1}(u) / U \text{ is open in } X\} \cup \{\pi_2^{-1}(v) / V \text{ is open in } Y\}$ is a sub-basis for the product topology on X × Y. **5**

2. A) Define a convex subset Y of an ordered set X. Prove that intervals and rays in X are convex in X, but converse is not true. **6**

   B) Let X be a topological space satisfying T_1 axiom and let A be a subset of X. Prove that the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A. **5**

   C) Give an example of a topological space which is not a Hausdorff space. Further, prove that a sequence of points of a Hausdorff space X converges to at most one point of X. **5**

3. A) State and prove the Pasting Lemma. Is the function f : [0, 1] ∪ [2, 3] → $\mathbb{R}$ defined by $f(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ x+1, & \text{if } x \in [2,3] \end{cases}$ continuous ? **6**

B) Find the closures of the sets $\mathbb{Z}$, $\mathbb{Q}$ and $\{Y_n | n = 1, 2, 3, ...\}$ in $\mathbb{R}$. **5**

C) Show that the subspace [a, b] of $\mathbb{R}$ is homeomorphic with [0, 1]. Further, show that [0, 1] is not homeomorphic with the subspace S^1 of $\mathbb{R}^2$. **5**

4. A) Prove that the topologies on $\mathbb{R}^2$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^2$. **6**

B) Let $f : \mathbb{R} \to \mathbb{R}^w$ be defined by F (t) = (t, t, t, ...). Prove that f is not continuous if $\mathbb{R}^w$ is given the box topology. **5**

C) Give an example of a quotient map which is not a closed map. **5**

5. A) Prove that a finite Cartesian product of connected spaces is connected. **6**

B) Prove that every path connected space is connected. Is converse true ? Justify your answer. **5**

C) What are components and path components of $\mathbb{R}_e$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_e$? **5**

6. A) Prove that a subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the Euclidean metric d or the square metric ρ. **6**

B) Show that if Y is compact, then the projection map $\pi_1 : X \times Y \to X$ is a closed map. **5**

C) Let (x, d) be a compact metric space. Let $f : X \to X$ be a function such that d (f(x), f(y)) = d (x, y) for all x, y $\in$ X. Show that f is a homeomorphism. **5**

7. A) Suppose that X has a countable basis, then prove that every open covering of X contains a countable subcollection covering X. **6**

B) Show that $\mathbb{R}_e$ and I_o^2 are not metrizable. **5**

C) Let f, g : X $\to$ Y be continuous maps. Suppose that Y is Hausdorff. Show that the set {x / f(x) = g (x)} is closed in X. **5**

8. A) Prove that every metrizable space is normal. **6**

B) Prove that a connected regular space having more than one point is uncountable. **5**

C) Show that a closed subspace of a normal space is normal. **5**

M.A./M.Sc. (Sem. – II) (2004 Pattern) Examination, 2010
MATHEMATICS
MT 601 : Real Analysis – II (Old)

Time : 3 Hours   Max. Marks : 80

*N.B.:1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) If $f : [a, b] \to \mathbb{R}$ is of bounded variation, then prove that f is also bounded and satisfies $\|f\|_\infty \leq |f(a)| + V_a^b f$. **8**

   b) Prove that BV [a, b] is complete under the norm $\|f\|_{BV} = |f(a)| + V_a^b f$. **8**

2. a) State Helly's first theorem and prove that $\|f_1 f_2\|_{BV} \leq \|f_1\|_{BV} \|f_2\|_{BV}$. **8**

   b) Give an example to show that "Every bounded function may not be Riemann - Stieltjes integrable". **8**

3. a) Prove that $C[a,b] \subset R\alpha\, [a,b]$ for any increasing α. **8**

   b) Suppose that α' exists and it is a bounded Riemann integrable function on [a, b]. Then show that given a bounded function 'f ' on [a, b]. We have, $f \in R_\alpha[a, b]$ if and only if $f\alpha' \in R[a,b]$, in either case $\int_a^b f\, d\alpha = \int_a^b f(x)\, \alpha'(x) dx$. **8**

4. a) If $f \in R_\alpha[a,b]$ with $m \leq f \leq M$, then show that $\int_a^b f\, d\alpha = C[\alpha(b) - \alpha(a)]$ for some 'C' between m and M and also if f is continuous then show that $C = f(x_0)$ for some x_0. **8**

   b) Prove that if $S_n \to S$, then $\sigma_n \to S$. **8**

5. a) Define Lebesgue outer measure and prove the following : **8**

   i) $0 \le m^*(E) \le \infty$, for any E.

   ii) If $E \subset F$, then $m^*(E) \le m^*(F)$.

   b) Prove that $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n)$ for any sequence (E_n) of subsets of $\mathbb{R}$. **8**

6. a) State and prove Lebesgue dominated convergence theorem. **8**

   b) Let $\{E_n\}$ be the sequence of measurable sets. Then prove that

   i) If $E_n \subset E_{n+1}$ for each n, then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. **6**

   c) State Vitali's covering theorem. **2**

7. a) Let $E \subset \mathbb{R}$, then prove that E is measurable iff

   $m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$ for every subset A of $\mathbb{R}$. **8**

   b) Let $\{f_n)$ be a sequence (finite or infinite) of measurable functions, then prove that $\sup_n f_n$ and $\inf_n f_n$ are measurable functions. **6**

   c) State Egorov's theorem. **2**

8. a) State and prove monotone convergence theorem. **8**

   b) Give an example of a improper Riemann integrable function which is not Lebesgue integrable. **8**

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[3721] – 204

M.A./M.Sc. Examination, 2010
MATHEMATICS
(2008 Pattern and 2004 Pattern
MT – 604 : Complex Analysis (New and Old)

Time: 3 Hours Max. Marks : 80

N.B. : 1) *Answer* ***any five*** *questions.*
2) *Figures to the* ***right*** *indicate* ***full*** *marks.*
3) $\mathbb{C}$ *and* $\mathbb{C}_\infty$ *denote complex plane and extended complex plane,* ***respectively****.*

1. a) If z and z' are points in the extended complex plane $\mathbb{C}_\infty$ and d(z, z') denote the distance between z and z' then derive the expression **8**

$$d(z,\ z') = \frac{2|z - z'|}{\left[(1+|z|^2)\,(1+|z'|^2)\right]^{\frac{1}{2}}}$$

b) i) For the point z = 3 + 2i, give the corresponding point of the unit sphere S in $\mathbb{R}^3$.

ii) Let z and z' be points in S (unit sphere in $\mathbb{R}^3$) corresponding to z and z' respectively. Let W be the point on S corresponding to z + z'. Find the coordinates of W in terms of the coordinates of z and z'. **8**

2. a) For a given power series $\sum_{n=0}^{\infty} a_n z^n$ define the number $0 \le R \le \infty$, by

$\frac{1}{R} = \lim\sup |a_n|^{\frac{1}{n}}$. Prove that **8**

i) If | z | < R, the series converges absolutely

ii) If | z | > R, the series diverges.

iii) If 0 < r < R then the series converges uniformly on { z : | z | $\le$ r }

b) Find the radius of convergence for each of the following power series **8**

i) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ ii) $\sum_{n=0}^{\infty} a^n z^n$, $a \in \mathbb{C}$

P.T.O.

3. a) Prove that if G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all z in G then f is constant. **8**

b) i) Show that for any z, $(\cos z)' = -\sin z$.

ii) Describe the set $\{z : e^z = -1\}$. **8**

4. a) If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and T is any Möbius transformation then prove that $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any point z_1. Hence prove that a Möbius transformation takes circles onto circles. **8**

b) i) Find the fixed points of a dilation and the inversion on $\mathbb{C}_\infty$.

ii) Evaluate the cross ratio $(7 + i, 1, 0, \infty)$. **8**

5. a) Prove that if a function f is analytic in the open sphere B (a ; R) then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z - a| < R \text{ where } a_n = \frac{1}{n!} f^{(n)}(a)$$ and this series has radius of convergence $\geq R$. **8**

b) Evaluate the following integrals

i) $\int_\gamma \frac{\sin z}{z^3} dz, \ \gamma(t) = e^{it}, \ 0 \leq t \leq 2\pi;$ **8**

ii) $\int_\gamma \frac{dz}{\left(z - \frac{1}{2}\right)^n}$ where n is a positive integer and $\gamma(t) = e^{it}, \ 0 \leq t \leq 2\pi$.

6. a) Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. Prove that if γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ then for a in $G - \{\gamma\}$

$$n(\gamma; a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz$$ **8**

b) i) Let γ be a closed rectifiable curve $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2 \int_\gamma (z-a)^{-n} dz = 0$

ii) Let p(z) be a polynomial of degree n and let R > 0 be sufficiently large so that p never vanishes in $\{z : |z| > R\}$. If $\gamma(t) = Re^{it}$, $0 \le t \le 2\pi$, show that

$$\int_\gamma \frac{p'(z)}{p(z)}\,dz = 2\pi i n \cdot$$ **8**

7. a) Let f be analytic in the region G except for the isolated singularities $a_1, a_2, \ldots, a_n$. Prove that if γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{n} n(\gamma; a_k)\,\text{Res}\,(f; a_k)$$ **6**

b) Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$; give the Laurent expansion of f (z) in the annuli ann (0 ; 1, 2). **5**

c) Show that for a > 1,

$$\int_0^\pi \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}.$$ **5**

8. a) Let G be a region in $\mathbb{C}$ and f an analytic function on G. Prove that if there is a constant M such that $\lim_{z \to a} \sup |f(z)| \le M$ for all a in $\partial_\infty G$ then $|f(z)| \le M$ for all z in G . **6**

b) Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G . Show that if there is a constant $c \ge 0$ such that | f(z) | = c for all z on the boundary of G then either f is a constant function or f has a zero in G . **5**

c) Does there exist an analytic function $f : D \to D$ with $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$? Justify your answer $\left(D = \{z : |z| < 1\}\right)$. **5**

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS (2008 Pattern)
MT-701 : Functional Analysis (New)

Time : 3 Hours Max. Marks : 80

Instructions : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Let M be a closed linear subspace of a normed linear space N. The norm of a coset x + M in the quotient space N/M is defined by

$\| x + M \| = \inf \{ \| x + m \| : m \in M\}$.

Prove that N/M is a normed linear space. **6**

b) Let $x = (x_1, x_2, ..., x_n)$ be an n-tuple of scalars. If $\|x\|_p = \left(\sum |x_i|^p\right)^{\frac{1}{p}}$, and $\|x\|_\infty = \max\{|x_1|,...,|x_n|\}$, then prove that $\| x \|_\infty = \lim \| x \|_p$, as $p \to \infty$. **4**

c) If M is a closed linear subspace of a normed linear space N, and if T is the natural mapping of N onto N/M defined by T (x) = x + M, show that T is a continuous linear transformation for which $\|T\| \leq 1$. **6**

2. a) Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + \{x_0\}$ is the linear subspace spanned by M and x_0, then prove that f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$. **8**

b) Let M be a linear subspace of a normed linear space N, and x_0 be a vector not in M. If d is the distance from x_0 to M, then show that there exists a functional f_0 in N* such that $f_0 (M) = 0$, $f_0 (x_0) = 1$, and $\|f_0\| = \frac{1}{d}$. **6**

c) True/False ? Justify your answer.

If N is complete, then N is reflexive. **2**

P.T.O.

3. a) State and prove the closed graph theorem. **8**

b) With usual notations prove that $x \to F_x$ is a norm preserving mapping of N into N^{**}. **8**

4. a) Show that the parallelogram law is not true in l_1^n $(n > 1)$. **4**

b) Let M be a proper closed linear subspace of a Hilbert space H. Prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$. **6**

c) Show that $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}$ is an orthonormal set in $L_2 [0, 2\pi]$. **6**

5. a) Prove that an operator T on a Hilbert space H is normal if and only if $\|T^* x\| = \|Tx\|$ for every $x \in H$. **6**

b) Show that an orthonormal set in a Hilbert space is linearly independent. **4**

c) Let P be a projection on a Hilbert space H with range M and null space N. Prove that $M \perp N$ if and only if P is self-adjoint. **6**

6. a) If T is an operator on a Hilbert space H, then prove that the following conditions are all equivalent to one another. :

i) $T^*T = I$;

ii) $(Tx, Ty) = (x, y)$ for all x and y;

iii) $\|Tx\| = \|x|$ for all x. **6**

b) Let N_1 and N_2 be normal operators on a Hilbert space H with the property that either commutes with the adjoint of the other. Prove that $N_1 + N_2$ and $N_1 N_2$ are normal. **6**

c) Prove that the adjoint operation $T \to T^*$ on B(H) has the following properties :

i) $(\alpha T)^* = \overline{\alpha} T^*$;

ii) $(T_1 T_2)^* = T_2^* T_1^*$. **4**

7. a) With usual notations, prove that $(l_p^n)^* = l_q^n$. **6**

b) Show that a projection on a Hilbert space H satisfies $O \le P \le I$. Under what conditions will P = O and P = I ? **6**

c) Let A and $A \subset B$ be nonempty subsets of a Hilbert space H. Show that

$A \subset A^{\perp\perp}$ and $B^{\perp} \subset A^{\perp}$. **4**

8. a) i) State spectral theorem.

ii) If T is a normal operator on a Hilbert space H, then prove that M_i's span H. **8**

b) Let T be an operator on H, and prove the following statements :

i) T is singular if and only if $0 \in \sigma(T)$;

ii) If T is non-singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$. **8**

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M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS (2004 Pattern)
MT-701 : General Topology (Old)

Time : 3 Hours Max. Marks : 80

N.B. : 1) Answer ***any five*** *questions.*
2) Figures to the ***right*** *indicate marks.*

1\. a) Define a basis for a topology on a set X. Show that the topology generated by a basis equals the collection of all unions of elements of the basis. **6**

b) Let X be a set and $\tau = \{u \subseteq x \mid x - u$ is a finite or all of x$\}$. Then show that τ is a topology on X. **5**

c) If X = { a, b, c}, let $\tau_1 = \{\phi, x, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, x, \{a\}, \{b, c\}\}$. Find the smallest topology containing τ_1 and τ_2, and the largest topology contained in τ_1 and τ_2. **5**

2\. a) Let A be a subset of the topological space X ; let A' be the set of all limit points of A. Then prove that $A = A \cup A'$. **6**

b) Is the real line $\mathbb{R}$ a Hausdorff space ? Justify. **5**

c) Find the closures of the following subsets of the real line IR ?

i) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}$

ii) The set Q of rational numbers. **5**

3\. a) Let $f : A \to X \times Y$ be given by the equation f (a) = (f_1 (a), f_2(a)). Prove that f is continuous if and only if the functions $f_1 : A \to X$ and $f_2 = A \to Y$ are continuous. **6**

b) Show that the mapping f : IR $\to$ IR given by f (x) = 3x + 1 is a homeomorphism. **5**

c) Suppose that f : X $\to$ Y is continuous. If x is a limit point of the subset A of X, is it necessarily true that f (x) is a limit point of f (A) ? **5**

4. a) Prove that a finite Cartesian product of connected spaces is connected. **6**

   b) Prove that the image of a connected space under a continuous map is connected. **5**

   c) Show that the set $\mathbb{R}^{w}$ is not collected in the box topology. **5**

5. a) Let Y be a subspace of X. Prove that Y is compact if and only if every covering of Y by sets open in X contains a finite sub collection covering Y. **6**

   b) Show that the real line $\mathbb{R}$ is not compact. **5**

   c) Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is closed map. **5**

6. a) Prove that if a topological space X has a countable basis then it is Lindelöf and separable. **6**

   b) Prove that the space $\mathbb{R}_e$ is first countable but not second countable. **5**

   c) Show by an example that the product of two Lindelöf spaces need not be Lindelöf. **5**

7. a) Prove that a subspace of a Hausdorff space is Hausdorff and a product of Hausdorff space is Hausdorff. **8**

   b) i) Show that a closed subspace of a normal space is normal.

   ii) Show that if πX_α is regular then so is X_α. **8**

8. a) Prove that every regular space X with a countable basis is metrizable. **10**

   b) State the Tychonoff Theorem. Hence show that the product $\prod_{n=1}^{\infty} [-n, n]$ is compact in the product topology. **6**

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS (2008 Pattern)
MT – 702 : Ring Theory (New)

Time: 3 Hours Max. Marks : 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate full marks.*

1. a) If R is a ring with identity and S is a subring of R containing the identity, then prove that if u is a unit in S then u is a unit in R, show by example that the converse is false. **5**

b) Define the ring of integers in the quadratic field $Q(\sqrt{D})$, D is square free integer.
Prove that the element α in ring of integers in the quadratic field is a unit iff norm of $\alpha = \pm 1$. **6**

c) i) Prove that the only Boolean ring that is an integral domain is $z/2z$. **3**

ii) If R is an integral domain and $x^2 = 1$ for some $x \in R$ then prove that $x = \pm 1$. **2**

2. a) If R is an integral domain and if $p(x), q(x) \in R[x]$ then prove that
i) degree p(x) q(x) = degree p(x) + degree q(x).
ii) R [x] is an integral domain. **6**

b) Find all ring homomorphisms from z to $\frac{z}{10z}$. Describe the kernel and image in each case. **6**

c) If $\phi : R \to S$ is a ring homomorphism and if x is nilpotent element of R then prove that $\phi(x)$ is a nilpotent of S. **4**

3. a) Prove that every ideal in a Euclidean domain is principal. **5**

b) If R is a quadratic integar ring $z[\sqrt{-5}]$ and $I = (3, 2 + \sqrt{-5})$, is an ideal then show that I is not principal ideal. Is R a Euclidean domain ? **6**

c) If R is a Euclidean domain and if a, b, c $\in R$ $(a \neq 0, b \neq 0)$ a divides bc then show that $\frac{a}{(a,b)}$ divides c. **5**

P.T.O.

4. a) Prove that every non-zero prime ideal in a principal ideal domain is a maximal ideal. Is Z[x] a principal ideal domain ? **6**

   b) Prove that a quotient of PID, in general, is not a PID; but quotient of by a prime ideal, ideal is PID. **6**

   c) Prove that the quotient ring $\frac{z[i]}{(1+i)}$ is a field of order 2. Is it a U.F.D. ? **4**

5. a) Prove that a polynomial of degree two or three over a field F is reducible iff it has a root in F. **5**

   b) If I is a proper ideal in the integral domain R and p(x) is a non constant monic polynomial in R [x]. If the image of p(x) in $\left(\frac{R}{I}\right)[x]$ cannot be factored in $\frac{R}{I}[x]$ into two polynomials of smaller degree then prove that p(x) is irreducible in R[x]. **6**

   c) Construct a field with nine elements. **5**

6. a) Show that the following are equivalent. **6**

      i) R is Noetherian ring.

      ii) Every non-empty set of ideals of R contains a maximal element under inclusion.

      iii) Every ideal of R is finitely generated.

   b) If the polynomial ring R[x] is Noetherian then prove that R is Noetherian. **4**

   c) Show that the ring of continuous real valued functions on [0, 1] is not a Noetherian ring. **6**

7. a) If I is an ideal in the commutative ring R then prove that rad I is an ideal containing I and $\frac{\text{rad } I}{I}$ is the nilradical of R/I . **8**

   b) Prove that in the ring of integers z, the ideal (a) is a radical ideal iff a is squarfree or zero. **4**

   c) Define affine algebraic set show that one point subsets of A^n for any n, affine n-space over the field k, are affine algebraic. **4**

8. a) If $J = J_{ac}R$ = Jacobson radical of R then prove that an element $x \in J$ iff $1 - rx$ is a unit for all $r \in R$. **6**

   b) Prove that Artirian integral domain is a field. **6**

   c) Prove that every PID is a Dedekind domain. **4**

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M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS (2004 Pattern)
MT – 702 : Mechanics (Old)

Time: 3 Hours Max. Marks : 80

*N.B. : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate full marks.*

1. a) Derive Lagrange's equations of motion using D'Alembert's principle. **6**

b) Write down the equations of constraints in cartesian co-ordinates for a small rigid rod of length l is allowed to move in any manner inside a balloon of fixed radius R > l, the end parts of the rod always touching the bolloon's surface. **5**

c) Find the equation of motion of a solid sphere rollig down on an incline using Lagrange multipliers for the rolling constraints. **5**

2. a) Explain the following terms :

i) Degree of freedom

ii) Generalized momentum

iii) Virtual work

iv) Cyclic co-ordinates. **6**

b) Show that the expression for the kinetic energy on the quadratic function of generalized velocities. **5**

c) If L is a Lagrangian for a system of n degree of freedom satisfying the Lagrange's equations, then show that $L^1 = L + \frac{dF}{dt}(a_1 ... a_n, t)$ also satisfies the Lagrange's equation, where F is any arbitrary, but differential function of its arguments. **5**

3. a) Set up the Lagrangian for two bodies moving under central force about their center of mass and show that it can be reduced to an equivalent one body problem. **6**

   b) Prove that angular momentum of a particle in central force field remains constant. **5**

   c) Find the central force under the action of which a particle will follow $r = a\,(1 + \cos\theta)$. **5**

4. a) Explain the following terms :

      i) Lagendre's Dual transformation

      ii) Passive variables. **5**

   b) Show that the Hamilton's principle

$$\delta \int_{\infty}^{t} L dt = 0$$

   also holds for the non-conservative system. **6**

   c) A particle moves on a smooth surface under gravity. Use Hamilton's principle to find the equation of motion. **5**

5. a) Deduce Newton's second law of motion from Hamilton's principle. **5**

   b) Prove that a co-ordinate which is cyclic in the Lagrangian is also cyclic in the Hamiltonian. **5**

   c) Find the Routhian for the Lagrangian

$$L = \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta\right)^2 + \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mgl \cos\theta$$

   Where I_1, I_3, m, g, l are constants. **6**

6. a) Explain the method to obtain the required canonical transform when generating function is given. **6**

   b) Show that the reflection about the $x_2\, x_3$ plane passing through the origin is canonical transform. Obtain its generating function. **5**

   c) Define Poission's bracket and show that it is invariant under canonical transformation. **5**

7. a) State and prove Jacobi-Poisson theorem on Poisson bracket. **5**

   b) Evaluate $[L_1, A_{jk}]$ and $[A_{jk}, A_{il}]$ where L = r × p and Aij = $x_i x_j + p_i\, p_j$. **6**

   c) Calculate the eigenvalues and eigen vector of the rotation matrix,

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ **5**

8. a) Prove the Jacobi's theorem for the time independent Hamilton – Jacobi theory. **5**

   b) Explain the method to find the complete integral of the Hamilton-Jacobi equation. **5**

   c) Consider the motion of a body of unit mass on the constrained path y = coshx under a potential $v = \frac{x^2}{2}$. Solve Hamilton's equation of motion directly as well as by using the Hamilton– Jacobi method. **6**

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M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT-704 : Measure and Integration (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*
iii) B denotes a σ-algebra of subsets of X and μ denotes a measure on (X, B).

1. A) Suppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$. Prove that there is a measurable function f such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \sim B_\alpha$. **6**

   B) If $E_1 \in \mathcal{B}$, $\mu E_1 < \infty$ and $E_i \supset E_{i+1}$, then prove that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n$. **5**

   C) Let $\langle f_n \rangle$ be a sequence of measurable functions that converges to a function f except at the points of set E of measure zero. Show that if μ is complete, then f is a measurable function. **5**

2. A) Let $\langle f_n \rangle$ be a sequence of non-negative measurable functions that converge almost everywhere on a set E to a function f. Prove that $\int_E f \leq \underline{\lim} \int_E f_n$. **8**

   B) If f and g are non-negative measurable functions and a and b are non-negative constants, then show that
   $\int af + bg = a\int f + b\int g$. **4**

   C) Give an example of a decreasing sequence $\langle \mu_n \rangle$ of measures on a measurable space such that the set function μ defined by $\mu E = \lim \mu_n E$ is not a measure. **4**

P.T.O.

3. A) Let ν be a signed measure on the measurable space (X, *B*). Prove that there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$. **6**

B) Show that if measures ν_1 and ν_2 are singular with respect to μ, then so is $c_1\nu_1 + c_2\nu_2$. **5**

C) Prove that every measurable subset of a positive set is itself positive. Further, prove that union of a countable collection of positive sets is positive. **5**

4. A) Let (X, *B*, μ) be a σ-finite measure space and ν a σ-finite measure defined on *B*. Then prove that we can find a measure ν_0, singular with respect to μ, and a measure ν_1, absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$. **6**

B) If $A \in \mathfrak{a}$ and if $\langle A_i \rangle$ is any sequence of sets in $\mathfrak{a}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$, prove that $\mu A \leq \sum_{i=1}^{\infty} \mu A_i$. **5**

C) Let (X, *B*, μ) be a finite measure space and g an integrable function such that for some constant M,

$\left|\int g\phi \, d\mu\right| \leq M \|\phi\|_{\perp}$ for all simple functions ϕ. Prove that $g \in L^{\infty}$. **5**

5. A) Let F be a bounded linear functional on $L^{P}(\mu)$ with $1 < p < \infty$. Show that there is a unique element $g \in L^{q}$ such that

$F(f) = \int fg \, d\mu$

and $\|F\| = \|g\|_{q}$, where $\frac{1}{p} + \frac{1}{q} = \infty$. **6**

B) Let X be a set consisting of two points. Construct an outer measure on X which is not regular. **5**

C) If μ is a finite Baire measure on the real line, then show that its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right. Further, show that $\lim_{x \to -\infty} F(x) = 0$. **5**

6. A) Let μ be a measure on a σ-algebra $\mathfrak{a}$ of subsets of X, and let *M* be a collection of subsets of X which is closed under countable unions and which has the property that for each $A \in \mathfrak{a}$ with $A \subset M \in \mathcal{M}$, we have $\mu A = 0$. Prove that there is an extension $\bar{\mu}$ to μ to the smallest σ-algebra *B* containing $\mathfrak{a}$ and *M* such that $\bar{\mu} M = 0$ for each $M \in \mathcal{M}$. **6**

B) Let B be a μ^* - measurable set with $\mu^* B < \infty$. Prove that $\mu_* B = \mu^* B$. **5**

C) Let A_i be a disjoint sequence of sets in $\mathfrak{a}$ Prove that

$$\mu_*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_*(E \cap A_i).$$ **5**

7. A) Let F be a closed subset of X. Prove that F is a locally compact Hausdorff space, and the Baire sets of F are those sets of the form $B \cap F$, where B is a Baire set in X. **6**

B) Let μ be a finite measure defined on a σ-algebra *M* which contains all the Baire sets of a locally compact space X. Prove that μ is regular if it is inner regular. **5**

C) Show that the intersection of two σ-compact sets is σ-compact. **5**

8. A) Let μ be a measure defined on a σ-algebra *M* containing the Baire sets. Assume either that μ is quasi regular or that μ is inner regular. If μ is outer regular for each compact set or if μ is inner regular for each bounded open set, then prove that μ is regular for each σ-bounded set in *M*. **8**

B) Let μ be a Baire measure on X. Prove that there are complete saturated measures $\bar{\mu}$ and $\underline{\mu}$ defined on a σ-algebra containing the Borel sets with $\bar{\mu}$ quasi regular, $\underline{\mu}$ inner regular, and $\bar{\mu} E = \underline{\mu} E = \mu E$ for each σ-bounded Baire set. **8**

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT-704 : Mathematical Methods – I (Old)
(2004 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Define conditionally convergent series and give an example of the same. **4**

b) Discuss convergence of the following series. **6**

i) $\sum_{n=1}^{\infty} n^4 e^{-n^2}$

ii) $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots$

c) Find first four terms of the Taylor series expansion of the function $\tan^{-1}x$ around x = 0. **4**

d) Explain the root test for convergence of a series. **2**

2. a) If $e=\sum_{n=0}^{\infty}\frac{1}{n!}$, show that 2 < e < 3. **4**

b) Show that the alternating series $a_1 - a_2 + a_3 - a_4 \ldots$, where $0 \le a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0,$ converges. **5**

c) State the Dirichlet conditions for convergence of Fourier series. **2**

d) Expand $f(x) = x^2$, $-\pi < x < \pi$ as Fourier series, where f is periodic with period π. **5**

3. a) Find the amplitude, period, frequency, wave velocity and wave length of the wave motion $y(x) = \sin\frac{5\pi x}{6}$. **5**

b) Define Legendre form of elliptic integrals of the first and second kind. **2**

c) Show that, if 0 < k < 1, the elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$= \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \ldots\right].$$ **5**

d) Find the length of the arc of the curve y = sinx, $0 \le x \le \pi$, in terms of elliptic integrals. **4**

4. a) Define $\Gamma(m)$ and β (m, n). Further show that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, m, n > 0. **6**

b) Evaluate $\int_0^{\pi/2} \sin^4\theta \cos^5\theta \, d\theta$. **4**

c) Prove the duplication formula

$$2^{2P-1}\,\Gamma(P)\,\Gamma\left(P+\tfrac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2P).$$ **6**

5. a) Show that

$\int_{-1}^{1} P_n(x)\,P_m(x)dx = 0$ if $m \ne n$, where P_n denotes Legendre polynomial. **4**

b) State the Rodrigues formula for Legendre polynomials. Evaluate $P_4(x)$ using the same. **4**

c) Show that for p = n(n + 1), n $\in \mathbb{N}$, Legendre equation $(1-x^2)y'' - 2xy' + py = 0$, admits a polynomial solution of degree n. **8**

6. a) Find the Laplace transform of : **6**

i) $L[e^{4t} \sinh 3t]\,(s)$

ii) $L\left[\frac{1-\cos t}{t}\right]$

b) Find inverse Laplace transform

$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right](t)$. **4**

c) Solve the following differential equation using the Laplace transform.
$y'' + 4y' + 8y = \cos 2t$, $y(0) = 2$, $y'(0) = 1$. **6**

7. a) State the Rodrigue's formula for Hermite polynomials and evaluate $H_2(x)$, $H_3(x)$. **4**

b) Solve the Bessel equation of order zero :

$x^2 y'' + xy' + x^2 y = 0$, around the regular singular point 0 and derive the expression for J_0. **8**

c) Show that $\frac{d}{dx} J_0(x) = -J_1(x)$. **4**

8. a) Define Fourier transform and prove that

i) $F\left[e^{iat} f(t)\right](s) = \hat{f}(s+a)$

ii) $F[f(t-a)]\,(s) = e^{ias}\,\hat{f}(s)$. **6**

b) Find Fourier transforms of

i) $f(t) = e^{-t^2}$

ii) $f(t) = e^{-|t|}$. **6**

c) State and prove Fourier convolution theorem. **4**

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS (2008 Pattern)
MT. 705 : Graph Theory (New)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Prove that if G is a self-complementary graph with n vertices, then n or n –1 is divisible by 4. **6**

b) Prove that an edge is a cut edge if and only if it belongs to no cycle. **6**

c) Prove that every set of six people contains at least three mutual acquaintances or three mutual strangers. **4**

2. a) Prove that if G is a simple n-vertex graph with $\delta(G) \geq \frac{(n-1)}{2}$, then G is connected. **6**

b) Prove that every simple graph with at least two vertices has at least two vertices of same degree. **6**

c) Prove that every Tournament has a king. **4**

3. a) Prove that for an n-vertex graph G (with n $\geq$ 1), the following are equivalent : **6**

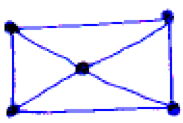
i) G is connected and has no cycles

ii) G is connected and has n – 1 edges

iii) G has n – 1 edges and no cycles.

b) Determine whether the sequence (5 5 5 4 2 1 1 1) is graphic ? Provide a construction or a proof of impossibility. **6**

c) Using matrix tree theorem, count the spanning trees in the graph G. **4**



4. a) Prove that in a connected weighted graph G, Kruskal's algorithm constructs a minimum weight spanning tree. **8**

   b) There are six cities in a network. The travel time for traveling directly from i to j is the entry a_{ij}, in the matrix below. Also, $a_{ij} = \infty$ indicates that there is no direct route. Determine the least travel time and quickest route from i to j for each pair i, j. **8**

$$\begin{pmatrix} 0 & 5 & \infty & 8 & 5 & 2 \\ 5 & 0 & 3 & 4 & \infty & 5 \\ \infty & 3 & 0 & 2 & 4 & \infty \\ 8 & 4 & 2 & 0 & 2 & 5 \\ 5 & \infty & 4 & 2 & 0 & 11 \\ 2 & 5 & \infty & 5 & 11 & 0 \end{pmatrix}$$

5. a) Prove that for k > 0, every k-regular bipartite graph has a perfect matching. **6**

   b) Define :

      i) Maximal matching in a graph

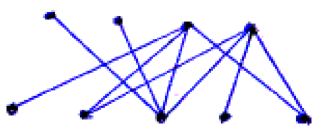
      ii) Maximum matching in a graph. **6**

      Find the smallest graph having a maximal matching that is not a maximum matching.

   c) Prove or disprove = Every tree has at most one perfect matching. **4**

6. a) Prove that if G is a graph without isolated vertices then $\alpha'(G) + \beta'(G) = n(G)$. **8**

   b) i) Find a maximum matching in the following graph.



      ii) Let T be a tree with n vertices, and let k be the maximum size of an independent set in T. Determine $\alpha'(T)$ in terms of n and k. **8**

7. a) Prove that if G is a 3 – regular graph then $k(G) = k'(G)$. **8**

   b) i) Determine $k(G)$, $k'(G)$ and $\delta(G)$ for the graph G where G is a complete graph on five vertices.

      ii) Show that every graph with connectivity 4 is 2-connected. **8**

8. a) Prove that a graph is 2-connected if and only if it has an ear decomposition. **8**

   b) i) State Menger's theorem. Illustrate with one example.

      ii) State Max-flow Min-cut theorem. Illustrate with one example. **8**

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS (2004 Pattern)
MT. 705 : Rings and Modules (Old)

Time : 3 Hours   Max. Marks : 80

*N.B.: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) If R is commutative ring with 1, then prove that $A \in M_n(R)$ is a unit iff its determinant det (A) is a unit in R. **6**

   b) If R is a ring with 1 and $x \in R$ is nilpotent then show that 1 + x is a unit in R. Can one replace 'nilpotent' by "zero divisor". **5**

   c) Is the following statement true ? Justify ? In the ring $Z_{2k,}\ \overline{k}$ is an idempotent if K is odd. **5**

2. a) If R is a ring with 1 and I is an ideal in R such that $I \neq R$ then prove that there is a maximal ideal M of the same kind as I such that $I \subseteq M$. **10**

   b) Show that the above result is not true if R has no unity even if R is commutative. **6**

3. a) Prove that the $Z/_{nZ}$ is a field iff $z/_{nz}$ is an integral domain or iff n is a prime. **8**

   b) If R is a commutative ring with unity and each ideal in R is prime then prove that R is a field. **4**

   c) If the intersection of two prime ideals is a prime ideal then prove that one of them is contained in the other. **4**

4. a) If for $n \geq 2$, the ring $z/_{nz}$ has no non-trivial nilpotent elements then prove that n is square tree. **6**

   b) Give an example of a ring in which an ideal of an ideal is not an ideal. **5**

   c) Show that in any Boolean ring an ideal is maximal iff it is a prime ideal. **5**

5\. a) If $I \subseteq J$ are both 2-sided ideals in a ring R then prove that $\dfrac{R/I}{J/I} \simeq R/J$. **8**

b) Give examples of homomorphisms of rings f : R $\rightarrow$ S and g : S $\rightarrow$ T such that $g_o f$ is an epimorphism but f is not. **4**

c) Prove that $\text{Hom}_{\text{rings}}(z_{n,} z) = (0) \quad \forall\, n \in N$. **4**

6\. a) Prove that a prime is an irreducible but not conversely. **8**

b) Prove that every Euclidean domain is a PID. **4**

c) Show that in the ring Z [i] the elements 3 + 4 i and 4 – 3 i are associates whereas $11 + 7i$ is co-prime to 18 – i. **4**

7\. a) If the ring R is an FD in which every irreducible element is a prime then prove that R is UFD. **5**

b) If R is UFD then prove that every irreducible polynomial in R [X] is a prime. **5**

c) i) Show by an example that a subring or a quotient of a UFD need not be a UFD. **3**

ii) Show by Eisenstein's criterion $x^2 + 1$ is irreducible over IR. **3**

8\. a) If M and N are submodules of a module P over R. Then prove that $M \cap N = (0) \Leftrightarrow$ every element $S \in M + N$ can be uniquely written as s = x + y with $x \in M$ and $y \in N$. **6**

b) Show that every finitely generated R-module M can be considered as a qualient of R^n for some n. **5**

c) Define Torsion module and torsion free module and give example for each. **5**

For any module M over a commutative integral domain R, prove that the quotient M/M_t is torsion free.

(M_t = set of all torsion elements of M).

M.A./M.Sc. Examination, 2010
MATHEMATICS (2005 Pattern)
MT-707 : Graph Theory (Old)

Time: 3 Hours Max. Marks: 80

***N.B.**: 1) Attempt **any five** questions.*

*2) Figures to the **right** indicate **full** marks.*

1. a) List all non-isomorphic simple directed graphs with three vertices. **6**

b) Prove that if G is bipartite, then every circuit in G has even length. **6**

c) If all vertices of a graph G have degree P, where P is an odd number, show that the number of edges in G is a multiple of P. **4**

2. a) If v and e denote the number of vertices and edges respectively in a connected planar graph G, with e > 1, then prove that $e \leq 3v - 6$. Hence, prove that K_5 is nonplanar. **8**

b) If a connected planar graph with n vertices, all of degree 3 has 7 regions, determine n. **4**

c) i) Find a planar graph that is isomorphic to its own dual. **4**

ii) For what values of r and s, is the complete bipartite graph $K_{r,s}$ planar ?

3. a) Prove that an undirected multigraph has an Euler Cycle if and only if it is connected and has all vertices of even degree. **8**

b) Find the chromatic number of Petersen's graph. Give justification. **4**

c) i) For which values of n, does K_n, the complete graph on n vertices have an Euler cycle ? **4**

ii) Prove or disprove: A graph with an Euler cycle have a bridge.

4. a) Prove that every tournament has a Hamilton path. **6**

b) Prove that every planar graph can be 5-coloured. **6**

c) Find the chromatic polynomial of the graph C_4, of a circuit of length 4. **4**

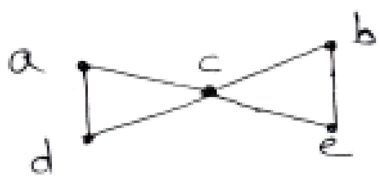
5. a) Prove that there are n^{n-2} different undirected trees on n lables. **6**

b) Show that any tree with more than one vertex has at least two vertices of degree one. **6**

c) Show that the chromatic polynomial of an n vertex tree in $K(K-1)^{n-1}$. **4**

6. a) Prove that Prim's algorithm yields a minimal spanning tree. **8**

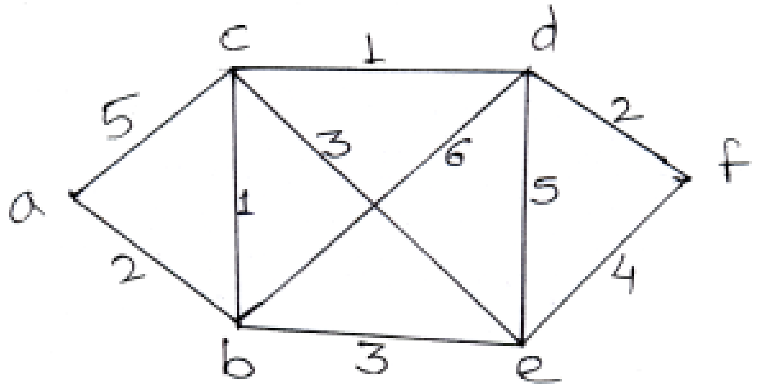
b) Find all spanning trees (upto isomorphism) in the graph G. **4**



c) If 56 people sign up for a tennis tournament, how many matches will be played in the tournament ? **4**

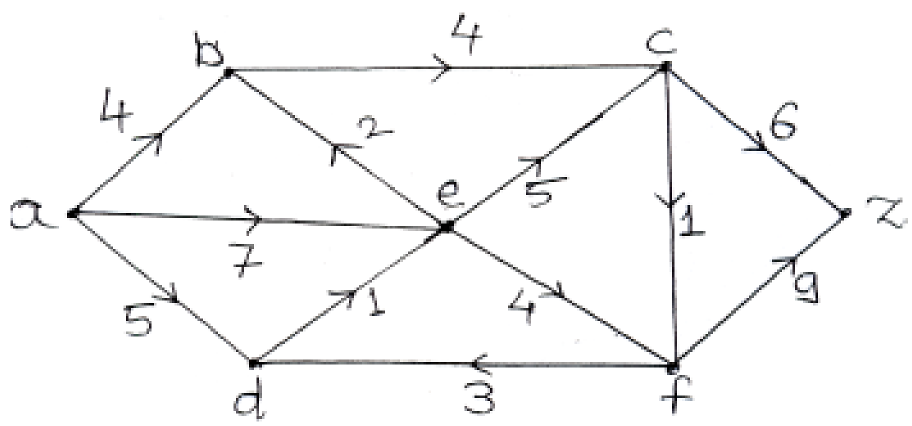
7. a) Prove that for any a – z flow f, and any a – z cut $(P, \overline{P})$, in a network N, $|f| \le K(P, \overline{P})$. **8**

b) Determine the shortest path from vertex a to f in the following graph, using Dijkstra's algorithm. **8**



8. a) State and prove Hall's marriage theorem. **8**

b) Find a maximal flow from a to z in the following Network. **8**



M.A./M.Sc. (Semester – IV) (2008 Pattern) Examination, 2010
MATHEMATICS
MT – 801 : Field Theory (New)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **marks**.*

1. a) Give an example of an irreducible polynomial in $\mathbb{Z}[x]$, having degree $n \geq 1$. Justify your answer. **4**

b) Let $F \subseteq E \subseteq K$ be fields. If $[K : E] < \infty$ and $[E : F] < \infty$, show that :

i) $[K : F] < \infty$ and

ii) $[K : F] = [K : E]\,[E : F]$. **6**

c) Let p (x) be an irreducible polynomial in F [x]. Show that there exists an extension E of F in which p (x) has a root. **6**

2. a) Show that a finite extension field is an algebraic extension. **4**

b) Let $E = F(u_1, ... u_n)$ be a finitely generated extension of F such that each u_i, i = 1,..., n is algebraic over F. Show that E is a finite extension of F and hence an algebraic extension of F. **6**

c) Let F be a field, and let $\sigma : F \to L$ be an embedding of F into an algebraically closed field L. Let $E = F(\alpha)$ be an algebraic extension of F. Show that σ can be extended to an embedding $\eta : E \to L$ and the number of such extensions is equal to the number of distinct roots of the minimal polynomial of α. **6**

3. a) Define the splitting field of a polynomial $f(x) \in F[x]$, where $\deg f(x) \geq 1$. **2**

b) Find the splitting field of $x^p - 1 \in Q[x]$' p odd prime, and also find the degree of the splitting field. **7**

c) Let E/F be an algebraic extension and suppose that every irreducible polynomial in F [x] that has a root in E splits into linear factors in E. Show that E is the splitting field of a family of polynomials in F [x]. **4**

d) Is $Q\left(2^{\frac{1}{3}}\right)$ a normal extension of Q ? Justify your answer. **3**

P.T.O.

4. a) If f $(x) \in F[x]$ is irreducible over F, then show that all roots of f(x) have the same multiplicity. **5**

   b) Show that if F is a finite field, the number of elements of F is p^n for some prime p and an integer $n \geq 1$. **5**

   c) Let p be a prime and n an integer ≥ 1. Show that the roots of $x^{p^n} - x \in \mathbb{Z}_p[x]$ in its splitting field are distinct and form a field F with p^n elements. Show also that F is the splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$. **6**

5. a) Suppose E is a finite separable extension of a field F. Show that E is a simple extension of F. **8**

   b) Let $F \subset E \subset K$ be three fields such that E is a finite separable extension of F and K is a finite separable extension of E.

   Show that K is a finite separable extension of F. **6**

   c) Is a $\mathbb{Q}(\sqrt{2})$ a separable extension of $\mathbb{Q}$? Why ? **2**

6. a) Let F and E be fields, let $\sigma_1, \sigma_2, ..., \sigma_n$ be distinct embeddings of F into E. Show that $\sigma_1, \sigma_2, ..., \sigma_n$ are linearly independent over E. **6**

   b) Let F be a finite normal separable extension of a field F. Show that F is the fixed field of G (E/F). **6**

   c) If E/F is a Galois extension and G (E/F) $\approx S_3$, find the number of intermediate fields between F and E. **4**

7. a) Prove that any polynomial of degree ≥ 1 in $\mathbb{C}[x]$ factorises into linear factors in $\mathbb{C}[x]$. **8**

   b) Let f $(x) \in F[x]$ and let E be the splitting field of f (x). Suppose G (E/F) is a solvable group. Show that f (x) is solvable by radicals over F. **8**

8. a) Show that the sum and difference of constructible numbers are constructible. **5**

   b) Show that it is impossible to construct a cube with volume equal to twice the volume of a given cube using ruler and compass only. **5**

   c) Show that the Galois group of $x^4 + 1 \in \mathbb{Q}[x]$ is the Klein four-group. **6**

M.A./M.Sc. (Semester – IV) (2004 Pattern) Examination, 2010
MATHEMATICS
MT – 801 : Algebraic Topology (Old)

Time : 3 Hours　　　　　　　　　　　　　　　　　　　　　　Max. Marks : 80

*N.B.: 1) Attempt **any five** questions.*
*2) **All** questions carry **equal** marks.*
*3) Figures to the **right** indicate **maximum** marks.*

1. a) When are two paths in a space X said to be path homotopic ? **4**

   b) Prove that path homotopy is an equivalence relation in the set of all paths in X. **8**

   c) Give an example of a space X, and two paths f, and g, in X, which start and end at the same points, such that : **4**

   i) f is homotopic to g　　　　ii) f is not homotopic to g.

2. a) Define the group $\Pi_1(X, x_0)$, and define the multiplication in this group. **4**

   b) Prove that $\Pi_1(\mathbb{R}^n, 0) = \{e\}$, the trivial group with one element. **6**

   c) Let $A \subseteq X$, and $r = X \longrightarrow A$ be a map such that r (a) = a for each $a \in A$. If $a_0 \in A$, prove that $r_A : \Pi_1(X, a_0) \longrightarrow \Pi_1(A, a_0)$ is surjective. **6**

3. a) Define a covering space, and give an example. **2**

   b) Define a universal covering space, and give an example. **4**

   c) Let $p : E \longrightarrow B$ be a covering map, let $p(e_0) = b_0$. Prove that any path $f : [0, 1] \longrightarrow B$, beginning at b_0, has a unique lifting to a path $\tilde{f} : [0,1] \longrightarrow E$, beginning at e_0. **4**

   d) If $g : S' \to S'$ is $g(z) = z^3$, calculate explicitly the map $g_* : \Pi_1(S', 1) \to \Pi_1(S', 1)$. **6**

4. a) Prove that there is no retraction of B^2 on to S'. **6**

   b) Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$, with $|a_{n-1}| + |a_{n-2}| + \dots + |a_1| + |a_0| < 1$.

   Prove that the equation $f(z) = 0$ has a root in the unit ball $B = \{z \in \mathbb{C} \setminus |z| < 1\}$. **6**

   c) Find the fundamental group of the space $B \times S'$, where $B = \{z \in \mathbb{C} \setminus |z| < 1\}$, $S' = \{z \in \mathbb{C} \setminus |z| = 1\}$. **4**

5. a) State the Seifert-van Kampen theorem. **4**

   b) Prove that if $n \geq 2$, the n-sphere S^n is simply connected. **6**

   c) i) Prove that $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n \neq 1$. **3**

   ii) Prove that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n \neq 2$. **3**

6. a) Prove that $\Pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\Pi_1(X, x_0) \times \Pi_1(Y, y_0)$. **6**

   b) Prove that $\Pi_1(P^2, y)$ is a group of order 2, where P^2 is the projective plane. **6**

   c) Let Y have the discrete topology, and $P : X \times Y \rightarrow X$ is $p(x, y) = x$. Prove that p is a covering map. **4**

7. a) Prove that the fundamental group of the figure eight is not abelian. **8**

   b) Let a and b be points of S^2, and A a compact space and let $f : A \rightarrow S^2 \setminus \{a, b\}$ be continuous. If a and b lie in the same component of $S^2 \setminus f(A)$, prove that f is null homotopic. **8**

8. a) State the Jordan Separation theorem. Define all the terms that you use. **4**

   b) Give an example of a space X and two closed curves Y_1 and Y_2 in X such that : **6**

   i) Y_1 separates X

   ii) Y_2 does not separate X.

   c) Let $p : E \longrightarrow B$, with E simply connected.

   Given any covering map $r : Y \longrightarrow B$, prove that there is a covering map $q : E \longrightarrow Y$ at $r_0 q = p$. **6**

M.A./M.Sc. Mathematics (2008 Pattern) (Sem. – IV) Examination, 2010
MT-803 : DIFFERENTIAL MANIFOLDS (New)

Time : 3 Hours Max. Marks : 80

***N.B.**: 1) Attempt **any five** questions.*
*2) **All** questions carry **equal** marks.*
*3) Figures to the **right** indicate **full** marks.*

1. a) Let W be a k-dimensional linear subspace of $\mathbb{R}^n$. Prove that there is an orthogonal transformation $h : \mathbb{R}^n \to \mathbb{R}^n$ that carries W onto the subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^n$. **8**

b) i) Define the volume of a parametrized manifold. **4**

ii) Prove that the volume of a parametrized manifold is invariant under reparametrisation. **4**

2. a) Define a k-manifold without boundary in $\mathbb{R}^n$. **4**

b) Let $f : \mathbb{R}^k \to \mathbb{R}$ be any differentiable function.

Prove that the graph of f, i.e. $G(f) = \left\{ (x, f(x)) \setminus x \in \mathbb{R}^k \right\}$ is a k-manifold without boundary in $\mathbb{R}^{k+1}$. **6**

c) Show that I = [0, 1] is a l – manifold in $\mathbb{R}'$. What is its boundary, ∂I ? **6**

3. a) Let f : $\mathbb{R}^n \to \mathbb{R}$ be C^2, $M = \left\{ x \in \mathbb{R}^n \mid f(x) = 0 \right\}$, $N = \left\{ x \in \mathbb{R}^n \mid f(x) \geq 0 \right\}$.

Suppose $M \neq \phi$, and D f(x) has rank 1 an each point of M. Prove that N is an n-manifold in $\mathbb{R}^n$ and $\partial N = M$. **8**

b) Find the area of a hemisphere of radius a > 0. **8**

4. a) Let T be a linear map between two vector spaces V and W;i.e. T : V $\to$ W.

i) Define the dual map T^*. **4**

ii) Prove that $T^* (f \otimes g) = T^* f \otimes T^* g$, where f and g are tensors on V. **4**

b) Give an example of an non zero alternating 2-tensor on $\mathbb{R}^3$. **4**

c) Let $\pi \in S_{k+l}$ be the permutation.

$$\pi = \begin{pmatrix} 1 & 2 & 3 \ldots k & k+1 \ldots k+l \\ k+1 & k+2 \ldots k+l & 1 & 2 \ldots k \end{pmatrix}.$$

Prove that sgn $\pi = (-1)^{kl}$. **4**

5. a) Let M be a k-manifold in $\mathbb{R}^n$, and $p \in M$,

   i) Define the tangent space to M at p, T_pM. **4**

   ii) Prove that T_pM is well defined. **4**

   b) Let $M = \left\{ x \in \mathbb{R}^3 \setminus x_1^2 + x_2^2 + x_3^2 = 1 \right\}$.

   Evaluate T_pM where $p = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$. **4**

   c) Let $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ be C^2. Prove that $\alpha_*(x; v)$ is the velocity vector of the curve $y(t) = \alpha(x + tv)$ corresponding to the parameter value t = 0. **4**

6. a) Let f : $\mathbb{R}^n \to \mathbb{R}$ be C^2

   i) Define the 1-form df (x) (x; v). **4**

   ii) Let f : $\mathbb{R}^3 \to \mathbb{R}$ be $f(x_1, x_2, x_3) = e^{x_1} . \sin (x_2x_3)$.
   Evaluate df (1, 2, 3) $((1, 2, 3); (4, 5, 6))$. **4**

   b) i) What is an exact form ? Give an example. **4**

   ii) What is a closed form ? Give an example. **4**

7. a) Let $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ be C^∞. If w is an *l*-form on $\mathbb{R}^n$, prove that
   $\alpha^* (dw) = d(\alpha^* w)$. **6**

   b) If $\alpha : \mathbb{R}^3 \to \mathbb{R}^6$ is C^∞, prove that
   $d\alpha_1 \wedge d\alpha_3 \wedge d\alpha_5 = (\det D\alpha \quad (1, 3, 5)) \; dx_1 \wedge dx_2 \wedge dx_3$. **4**

   c) Let A = $(0, 1)^3$, $\alpha : A \to \mathbb{R}^4$ is $\alpha(s, t, u) = \left(s, u, t, (2u - t)^2\right)$, $Y_\alpha = \alpha(A)$.

   Evaluate $\int_{Y_\alpha} x_1 dx_1 \wedge dx_4 \wedge dx_3 + 2x_2x_3 \, dx_1 \wedge dx_2 \wedge dx_3$. **6**

8. a) When is a manifold said to be orientable ? **4**

   b) Give an example of a orientable manifold. Justify your answer. **4**

   c) Prove that any n-manifold in $\mathbb{R}^n$ is an oriented manifold. **4**

   d) State the generalised Stokes theorem. Define all the terms that you use. **4**

[3721] – 403

M.A./M.Sc. Mathematics (2004 Pattern) (Sem. – IV) Examination, 2010
MT-803 : MEASURE AND INTEGRATION (Old)

Time : 3 Hours   Max. Marks : 80

***N.B.**: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*
3) $\mathcal{B}$ denotes σ-algebra of subsets of X, μ denotes measure on the measure space $(X, \mathcal{B})$.

1. a) Suppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for $\alpha < \beta$. Prove that there is a measurable function f such that $f \le \alpha$ a.e. on B_α and $f \ge \alpha$ a.e. on $X \sim B_\alpha$. **6**

   b) If $E_i \in \mathcal{B}$ for i = 1, 2,..., then prove that $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu E_i$. **5**

   c) Show that if μ is complete and $E_1 \in \mathcal{B}$ and $\mu(E_1 \Delta E_2) = 0$, then $E_2 \in \mathcal{B}$. **5**

2. a) Let $(X, \mathcal{B})$ be a measurable space, $\{u_n\}$ a sequence of measures that converge setwise to a measure μ, and $\{f_n\}$ a sequence of non-negative measurable functions that converge pointwise to the function f.
   Prove that $\int f d\mu \le \underline{\lim} \int f_n \, d\mu_n$. **8**

   b) State and prove Monotone convergence theorem. **4**

   c) Prove that the union of a countable collection of positive set is positive. **4**

3. a) Let f be an extended real-valued function defined on X. Then prove that the following statements are equivalent :

   i) $\{x : f(x) < \alpha\} \in \mathcal{B} \; \forall \alpha$ ii) $\{x : f(x) \le \alpha\} \in \mathcal{B} \; \forall \alpha$

   iii) $\{x : f(x) > \alpha\} \in \mathcal{B} \; \forall \alpha$ iv) $\{x : f(x) \ge \alpha\} \in \mathcal{B} \; \forall \alpha$. **6**

   b) If f and g are non-negative measurable functions and a, b are non-negative constants, prove that $\int af + bg = a\int f + b\int g$. **5**

   c) If v_1 and v_2 are any two signed measures, then prove that $\alpha v_1 / \beta v_2$ is signed measure, where α, β are real numbers. **5**

4. a) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and v a σ-finite measure defined on $\mathcal{B}$. Then prove that there is a measure v_0, singular with respect to μ and a measure v_1, absolutely continuous with respect to μ such that $v = v_0 + v_1$. **6**

   b) Let $(X, \mathcal{B}, \mu)$ be a finite measure space and g be an integrable function such that for some constant M.
   $\left| \int g\phi d\mu \right| \le M \| \phi \|_p$ for all simple functions ϕ. Prove that $g \in L^2$. **5**

   c) If v is a signed measure such that $v \perp \mu$ and $v << \mu$, prove that v = 0. **5**

5. a) Let μ be a measure on an algebra a, μ^* the outer measure induced by μ and E any set. Prove that for $\varepsilon > 0$, there is a set $A \in a$ with $E \subseteq A$ and $\mu^* A \le \mu^* E + \varepsilon$. Also there is a set $B \in a_{\sigma\delta}$ with $E \subseteq B$ and $\mu^* E = \mu^* B$. **6**

b) Prove that the set function μ^* is an outer measure. **5**

c) Let $\{(A_i \times B_i)\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$. Prove that $\lambda(A \times B) = \Sigma\lambda(A_i \times B_i)$. **5**

6. a) Let E and F be disjoint sets. Show that

$\mu_* E + \mu_* F \le \mu_*(E \cup F) \le \mu_* E + \mu^* F \le \mu^*(E \cup F) \le \mu^* E + \mu^* F$. **6**

b) By assuming $\mu_* E \le \mu^* E$ and $E \in a$ prove that $\mu_* E = \mu E = \mu^* E$.. **5**

c) Let B be a μ^* -measurable set with $\mu^* B < \infty$. Prove that $\mu_* B = \mu^* B$. **5**

7. a) Let μ^* be a topologically regular outer measure on X. Prove that each Borel set is μ^*-measurable. **6**

b) Let μ be a finite measure defined on a σ-algebra m which contains all the Baire sets of a locally compact space X. If μ is inner regular, prove that it is regular. **5**

c) Let K be a compact set, O an open set with $K \subseteq O$. Prove that $K \subseteq U \subseteq H \subseteq O$. where U is a σ-compact open set and H is a compact G_δ. **5**

8. a) Let F be a closed subset of X topological space. Then F is a locally compact Hausdorff space and the Baire sets of F are those sets of the form $B \cap F$, where B is a Baire set in X. **6**

b) Let $\bar{\mu}$ be a nonnegative extended real valued function defined on the class of open subsets of X and satisfying

i) $\bar{\mu} O < \infty$, if $\bar{O}$ is compact   ii) $\bar{\mu} O_1 \le \bar{\mu} O_2$, if $O_1 \subseteq O_2$

iii) $\bar{\mu}(O_1 \cup O_2) = \bar{\mu} O_1 + \bar{\mu} O_2$, if $O_1 \cap O_2 = \phi$ iv) $\bar{\mu}(UO_i) \le \sum_i \mu O_i$

v) $\bar{\mu}(O) = \sup \{\mu U \mid \bar{U} \subseteq O, \bar{U} \text{ is compact}\}$

Prove that set function μ^* defined by $\mu^* E = \inf\{\bar{\mu} O : E \subseteq O\}$ is a topologically regular outer measure. **6**

c) Prove that every σ-bounded set E is contained in a σ-compact open set O. **4**

M.A./M.Sc. (Sem. – IV) Mathematics (2008 Pattern) Examination, 2010
MT 804 : ALGEBRAIC TOPOLOGY (New)

Time : 3 Hours Max. Marks : 80

***N.B.**: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Give an example of a covariant function. **4**

b) Let $i: S^{n-1} \to B^n$ be the inclusion map, and $I: S^{n-1} \to S^{n-1}$ be the identity. Prove that there exists $f: B^n \to S^{n-1}$ with $f \circ i = I$, if and only if the identity map I is homotopic to a constant map. **8**

c) i) Define a strong deformation retract. **4**

ii) Give an example of a strong deformation retract.

2. a) Let $A \subseteq X$. Prove that the relation of being homotopic relative to A is an equivalence relation. **4**

b) Let f, g : $X \to S^n$ be continuous mappings such that $f(x) + g(x) \neq 0 \ \forall x \in X$. Prove that f is homotopic to g. **4**

c) i) When is a space said to be contractible ? **2**

ii) Give an example of a space that is contractible. **2**

iii) Give an example of a space that is not contractible. **4**

3. a) If f is any path in X, and g is a null path in X such that $f * g$ exists, prove that $f * g$ and f are equivalent. **4**

b) Give an example to two paths f and g between two points x_0 and x_1 in a space X which are not equivalent. **6**

c) Let $x_0, x_1 \in X$, where X is path connected. Prove that $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. **6**

4. a) If A is a strong deformation retract of X, show that the inclusion map $i: A \to X$ induces an isomorphism $i^*: \pi_i(A, a) \to \pi_1(X, a)$ for any point $a \in A$. **4**

P.T.O.

b) Prove that a contractible space has a trivial fundamental group. **4**

c) i) If X and Y are homeomorphic, and path connected prove that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic. **4**

ii) Is the converse true ? **4**

5. a) Define the higher homotopy groups $\pi_n(X, x_0)$. **4**

b) Prove that every non-constant complex polynomial has a root in complex numbers. **8**

c) Draw a torus and calculate its fundamental group. **4**

6. a) i) Define a covering map. **2**

ii) Give an example of a covering map. **2**

b) Prove that a covering map is α local homeomorphism. **4**

c) Let G be a group acting on a space X. When is the action of G on X said to be properly discontinuous ? Give an example. **8**

7. a) Define a fibration, and give an example of a fibration. **4**

b) Let $p : \tilde{X} \to X$ be a fibration with unique path lifting. Suppose that f and g are paths in $\tilde{X}$ with f(0) = g(0), and pf ~ pg, prove that f ~ g. **6**

c) i) Find the fundamental group of $\mathbb{R}^2 \setminus \{0\}$. **3**

ii) Is $\mathbb{R}^1$ homeomorphic to $\mathbb{R}^2$? **3**

8. a) When is a set of points in $\mathbb{R}^n$ said to be geometrically independent ? Give an example. **4**

b) Define the boundary $\partial_p C_p$ of a p-chain C_p. **6**

c) Prove that the boundary of the boundary of a p-chain is zero. **6**

M.A./M.Sc. (Sem. – IV) Mathematics (2004 Pattern) Examination, 2010
MT 804 : MATHEMATICAL METHODS – II (Old)

Time : 3 Hours | Max. Marks : 80

*N.B.: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Solve the non-homogeneous Fredholm integral equation **8**

$$u(x) = x + \lambda \int_0^1 (xt^2 + x^2 t)\, u(t)\, dt.$$

b) Find the eigenvalues of the homogeneous Fredholm equation with degenerate Kernel

$$u(x) = \lambda \int_0^{\pi} \left[\cos^2 x \cos 2t + \cos 3x \cos^3 t\right] u(t) dt .$$ **8**

2. a) Prove that eigenvalues of a real symmetric kernel are real. **5**

b Show that eigen functions of a symmetric kernel corresponding to different eigenvalues are orthogonal. **6**

c) The multiplicity of any non-zero eigenvalue is finite, when

$$\int_a^b \int_a^b | k(x, t) |^2 dxdt < \infty, \text{ where } k(x, t) = k(t, x).$$ **5**

3. a) Prove that every continuous function g(s) defined by $g(s) = \int k(s,t)\, h(t)\, dt$ where k(s, t) is symmetric kernel, can be expanded as a series of eigen functions of k(s, t). **8**

b) Find Neumann series solution for the integral equation **8**

$$u(x) = f(x) + \lambda \int_0^1 xe^t u(t) dt.$$

4. a) In the light of Fredholm alternative discuss the existence of solutions to the non-homogeneous Fredholm equation

$$u(x) = f(x) + \lambda \int_0^{\pi} \left[\cos^2 x \cos 2t + \cos 3x \cos^3 t\right] u(t)\, dt .$$ **8**

b) Find the resolvent kernel of the integral equation

$$u(x) = 1 + \lambda \int_0^1 (1 - 3xt)\, u(t)\, dt \,. \qquad \textbf{8}$$

5. a) Find the curve with fixed end points such that its rotation about x-axis gives rise to a surface of minimum surface area. **8**

b) Determine the extremal of the functional $I[y(x)] = \int_{-l}^{l} \left[\frac{1}{2}\mu y''(x) + \rho y(x) \right] dx$

subject to $y(-l) = y(l) = y'(-l) = y'(l) = 0$. Here, μ, ρ are given constants. **8**

6. a) Find the extremals of the functional

$$I = \int_{x_1}^{x_2} (2yz - 2y^2 - y'^2 - z'^2)dx \,. \qquad \textbf{8}$$

b) Find the curve of fixed length L > 1, joining the points (0, 0) and (1, 0) in the plane that lies above the x-axis and encloses the maximum area between itself and the x-axis. **8**

7. a) Show that if y(x) is a piecewise continuous function and $\int_{x_0}^{x_1} y(x)\eta(x) = 0$,

holds for arbitrary continuous fuinctions $\eta(x)$ satisfying the conditon :

$\int_{x_0}^{x_1} \eta(x) = 0$ then y(x) is a constant. **4**

b) Explain the Legendre condition. **4**

c) Find the curve joining given points A and B which is traversed by a particle moving under gravity from A and B in the shortest time. (This is known as the Brachistochrone problem.) **8**

8. a) Show that the triangle with greatest area A for a given perimeter is equilateral. **8**

b) Find geodesics on a unit sphere. **8**

M.A./M.Sc. Examination, 2010
MATHEMATICS (2005 Pattern)
MT 806 : Lattice Theory (Old)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Prove that the set A of all real valued functions defined on X : for f, g $\in$ A, set f$\leq$g if and only if f (x) $\leq$ g (x) for all x $\in$ X is a lattice. **5**

b) Let < p ; $\leq$ > be a post in which inf H exists for all H$\subseteq$ P. Show that < p ; $\leq$ > is a lattice. **5**

c) Prove that I is a prime ideal of a lattice L if and only if there is a homomorphism Q of L onto C_2 with I = $Q^{-1}\{0\}$. **6**

2. a) Let L and K be lattices, let θ and $\oint$ be congrence relations of L and K respectively. Define the relation $\theta\times\oint$ on $L\times K$ by $<a,b>\equiv<c,d>(\theta\times\theta\times\oint)$ if and only if $a\equiv c(\theta)$ and $b\equiv d\,(\oint)$. Then show that $\theta\times\oint$ is a congruence relation on $L\times K$ and conversely, every congruence relation of $L\times K$ is of this form. **8**

b) Prove that dual of a distributive lattice is distributive. **4**

c) Prove that if a lattice L is finite then L and $\mathrm{Id}(L)$, the ideal lattice of L, are isomorphic. **4**

3. a) Let L be a lattice and Con (L) be the set of all its congruences. Then prove that Con (L) is a lattice. **6**

b) State and prove Nachbin Theorem. **8**

c) Show that $N_S \cong L\times K$ implies that L or K has only one element. **2**

4. a) Prove that a lattice is modular if and only if it does not contain a pentagon. **8**

   b) State and prove Hashimoto theorem. **8**

5. a) Let L be a lattice of finite length. If L is semimodular then prove that any two maximal chains of L are of same length. **8**

   b) Let L be semimodular lattice. Prove that if p and q are atoms of L, $a \in L$ and $a < a \vee q \leq a \vee p$, then prove that $a \vee p = a \vee q$. **4**

   c) Let L be a lattice of finite length. If L satisfies the condition : a, b $\in$ L with $a \neq b$, a and b cover $a \wedge b$, then $a \vee b$ covers a and b. Then prove that L is semimodular. **4**

6. a) Let L be a lattice and a, b $\in$ L. Then prove that the following conditions are equivalent. **8**

   i) a M b (i.e. (a, b) is a modular pair)

   ii) $\psi_b : x \rightarrow x \wedge b$, $x \in [a, a \vee b]$ is onto.

   iii) $Q_a : y \rightarrow y \vee a$, $y \in [a \wedge b, a]$ is one to one.

   b) Let L be a distributive lattice, I be an ideal and D be a dual ideal of L such that $I \cap D = Q$. Then prove that there exists a prime P such that $I \subseteq P$ and $P \cap D = Q$. **8**

7. a) Prove that a lattice L is Boolean if and only if it is isomorphic to some field of sets. **7**

   b) Prove that a lattice L is conditionally complete, if every bounded non-empty subset of L has g.*l*.b. **5**

   c) Illustrate with an example that the ideals of a Boolean lattice do not form a Boolean lattice. **4**

8. a) Define an isotone function f on a lattice L into L and prove that if L is a complete lattice and f is an isotone function on L into L then f (a) = a for some a $\in$ L. **8**

   b) If L is a finite Boolean lattice then prove that the ideal lattice Id (L) of L is Boolean. **5**

   c) Prove that any modular lattice can be embedded in a complete modular lattice. **3**

[3721] – 203

M.A./M.Sc. (Sem. – II) (2008 Pattern) Examination, 2010
MATHEMATICS
MT-603 : Groups and Rings (New)

Time : 3 Hours   Max. Marks : 80

N.B.: *i) Attempt **any five** questions.*
*ii) Figures to **right** indicate **full** marks.*

1. a) If G = (a) is a cyclic group of order n, generated by a, then prove that for each positive divisor k of n, the group G has exactly one subgroup of order k namely $\left(a^{\frac{n}{k}}\right)$. **6**

   b) i) If a group G contains elements a and b such that | a | = 4, | b | = 2 and $a^3b = ba$, then find | ab |. **3**

   ii) Show that $U(10) \not\approx U(8)$. **2**

   c) If the group G is with exactly eight elements of order 10, how many cyclic subgroups of order 10 does G have ? Is G cyclic ? **5**

2. a) If the pair of cycles $\alpha = (\alpha_{1,} \ldots, \alpha_m)$ and $\beta = (\beta_{1,} \ldots,\ldots, \beta_n)$ have no entries in common, then prove that $\alpha\beta = \beta\alpha$. **5**

   b) i) What are possible orders for the elements of S_6 and A_6 ? **6**

   ii) What is the maximum order of any element in S_{10} ?

   c) i) Find two groups H and K such that H $\neq$ K but, Aut(H) $\simeq$ Aut(K).

   ii) Find Aut(Z). **5**

3. a) State and prove Lagrange's theorem for finite groups. Is the converse of Lagrange's theorem true ? Justify. **8**

   b) i) If a group G contains elements of orders 1 through 10, what is the minimum possible order of G ? **8**

   ii) Show that in a group G of odd order, the equation x^2 = a has a unique solution for all a in G.

P.T.O.

4. a) If G and H are two finite cyclic groups, then prove that $G \oplus H$ is cyclic iff $|G|$ and $|H|$ are co-prime. **6**

b) If $G = \{e, x, x^2, y, yx, yx^2\}$ is a non-abelian group with $|x| = 3$, $|y| = 2$ then prove that $xy = yx^2$. **5**

c) If G is a non-abelian group of order p^3, p is a prime, and $Z(G) \neq \{e\}$ then prove that $|Z(G)| = p$. **5**

5. a) If ϕ is a group homomorphism from a group G to $\overline{G}$ with kernel ϕ as K, then prove that $\frac{G}{K} \simeq \phi(G)$. **5**

b) Determine all homomorphisms from Z_6 to Z_{15}. **6**

c) Find all abelian groups (upto an isomorphism) of order 360. **5**

6. a) Suppose that G is a finite abelian group of order $p^n m$ where p is a prime that does not divide m, then prove that $G = H \times K$ where $H = \left\{x \in G \mid x^{p^n} = e\right\}$ and $K = \left\{x \in G \mid x^m = e\right\}$. Also show that $|H| = p^n$. **6**

b) What is the smallest positive integer n such that there are two nonisomorphic groups of order n ? **5**

c) Calculate the number of elements of order 2 in the group Z_{16}. **5**

7. a) If G is a finite group and p is a prime such that p^k divides $|G|$, then prove that G has at least one subgroup of order p^k. **6**

b) Use Sylow's theorem to prove that any group of order 99 is isomorphic to Z_{99} or $Z_9 \oplus Z_{11}$. **5**

c) Calculate all conjugacy classes for quaternian group Q_8. **5**

8. a) Prove that if H is a subgroup of a finite group G and $|H|$ is a power of a prime p then H is contained in some Sylow p-subgroup of G. **6**

b) Find all the Sylow Z-subgroups of S_3. **5**

c) Suppose that G is a group of order 48, show that the intersection of any two distinct Sylow 2-subgroups of G has order 8. **5**

[3721] – 203

M.A./M.Sc. (Sem. – II) (2004 Pattern) Examination, 2010
MATHEMATICS
MT-603 : Group Theory (Old)

Time : 3 Hours Max. Marks : 80

N.B.: i) Attempt ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) If ϕ: is a homomorphism of the group G into the group G', then prove that
$\phi(1)=1$
$\phi(x^n)=(\phi(x))^n \ \forall x \in G, n \in Z$. **5**

b) Prove that no two of the additive groups Z, Q, IR are isomorphic to each other. **6**

c) Show that for $n \geq 2$, the (n – 1) transpositions (12) (23) ... . (n – 1 n) generates S_n. **5**

2. a) If m and n are integers, not both zero, then prove that the subgroup < m, n > of Z generated by them is the cyclic subgroup generated by their g.c.d. **6**

b) Determine the orders of all elements of S_4. **5**

c) If G has trivial centre, then show that for $a \neq b$ in G, the inner automorphisms j_a and j_b are distinct. Deduce that S_3 has at least six distinct inner auto-morphisms. **5**

3. a) If G is a finite group of order n such that for every divisor d of n, G has at most one subgroup of order d, then prove that G is cyclic. **6**

b) Prove that the converse of Lagrange's theorem holds in S_4 but does not hold in A_4. **8**

c) If $G = S_3$ and $H = \langle(2\ 3)\rangle$, find $x \in S_3$ such that $xH \neq Hx$. **2**

4. a) Prove that a group of order p^n, p is a prime and $n \geq 1$ has non-trivial centre. **5**

b) Find all conjugacy classes of Q_8, quaternian group, and hence write its class equation. **6**

c) i) Show that SL (n, z) Δ GL (n, z). **3**

ii) In any group G show that ab and ba are conjugate to each other. **2**

5. a) If H and K are subgroups of G, at least one being normal in G, then prove that HK = KH is a subgroup of G. What happens if both are normal subgroups ? **6**

b) Show that the Klein's four group V_4 is a normal subgroup of S_4. Find $\frac{S_4}{V_4}$. **6**

c) Prove that a finite abelian group of square free order is cyclic. **4**

6. a) If $\phi : G \to G'$ is a surjective homomorphism with Kernal N, then prove that $\frac{G}{N} \simeq G'$. **5**

b) If T is the multiplicative group of complex numbers of absolute value 1 then show that $\frac{\mathbb{R}}{Z} \simeq T$. **6**

c) If G acts on the set X, then show that for $s \in G, x \in X$, stab (sx) = $s(\text{stab}(x))s^{-1}$. **5**

7. a) If the prime power p^k divides the order n of a finite group G then prove that G contains a subgroup of order p^k. **6**

b) Prove or disprove any group of order 33 is cyclic. **5**

c) Find the number of elements of order five in a group of order 25. **5**

8. a) If a finite group G of order $n = kl$, $(k, l) = 1$, has normal subgroups A and B of orders k, l respectively then prove that G = AB (direct). **5**

b) If H Δ G and if H and $\frac{G}{H}$ are both soluble, then prove that G is soluble. **6**

c) Prove or disprove A group of order 200 is soluble. **5**

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M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS (2008 Pattern)
MT-802 : Combinatorics (New)

Time : 3 Hours Max. Marks : 80

***N.B.**: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. A) How many sequences of length 5 can be formed using the digits 0, 1, 2, ..., 8, 9 with and without repeatation ? Also find the number of sequences of length 5 that can be formed using the digits 0, 1, 2, ... , 8, 9 with the property that exactly two of the ten digits appear (Eg. : 00550). **6**

   B) How many arrangements of the seven letters in the word "SYSTEMS" have the E occurring somewhere before the M ? How many arrangements have E somewhere before the M and the three 'S's grouped consecutively ? **6**

   C) What is the probability that 2 (or more) people in a random group of 25 people have a common birthday ? **4**

2. A) Among all arrangements of "WISCONSIN" without any pair of consecutive vowels, what fraction have W adjacent to an I ? **6**

   B) How many integer solution are there to the equation $x_1 + x_2 + x_3 + x_4 = 30$, with $x_i \geq 0$? How many solutions with $x_i \geq i$? How many solutions with $x_1 \geq 2, x_2 \geq 2, x_3 \geq 4, x_4 \geq 1$? **6**

   C) Use generating functions to find the number of ways to collect \$ 15 from 20 distinct people if each of the first 19 people can give a dollar (or nothing) and twentieth person can give either \$ 1 or \$ 5 or nothing. **4**

3. A) Using summation method find a generating function for $a_r = r\,(r + 2)$. **6**

   B) Prove by combinatorial argument that $C(n, 1) + 6C(n, 2) + 6C(n, 3) = n^3$ and evaluate $1^3 + 2^3 + ... + (n - 1)^3 + n^3 = ?$ **6**

   C) Find the number of r-digit quaternary sequences with an even number of 0's and odd number of 1's. **4**

4. A) State and prove Burnside's theorem. **8**

B) Use generating functions to the set of simultaneous recurrence relations given below

$a_n = a_{n-1} + b_{n-1} + c_{n-1}$, $b_n = 3^{n-1} - c_{n-1}$

$c_n = 3^{n-1} - b_{n-1}$, $a_1 = 1 = b_1 = c_1$. **8**

5. A) State and prove the Inclusion-Exclusion formula. **6**

B) Solve the recurrence relation

$a_n = a_1 a_{n-1} + a_2 a_{n-2} + \ldots + a_{n-1} a_1$

where $a_0 = 0$ and $a_1 = 1$. **6**

C) Find the coefficient of x^{25} in $(1 + x^3 + x^8)^{10}$. **4**

6. A) How many different 3-colorings of the bands of an n hand baton are there, if the baton is unoriented ? **6**

B) Find the pattern inventory of black-white edge colouring of a tetrahedron. **6**

C) Find the number 7 bead necklaces distinct under rotations using 3 black and 4 white beads. **4**

7. A) How many ways are there to send six different birthday cards denoted $C_1, C_2, C_3, C_4, C_5, C_6$ to three aunts and three uncles, denoted $A_1, A_2, A_3, U_1, U_2, U_3$ if aunt A_1 would not like cards C_2 and C_4 ; if A_2 would not like C_1 or C_5 ; if A_3 likes all cards ; if U_1 would not like C_1 or C_5 ; if U_2 would not like C_4 ; and if U_3 would not like C_6 ? **6**

B) Find the exponential generating function for the number of ways to place r (distinct) people into three different rooms with at least one person in each room. Repeat with an even number of people in each room. **6**

C) Using combinatorial argument, prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$. **4**

8. A) How many ways are there to select 25 toys from seven types of toys with between two and six of each type ? **6**

B) Solve the following recurrence relations when $a_0 = 1$.

i) $a_n^2 = 2a_{n-1}^2 + 1$ ii) $a_n = -n\, a_{n-1} + n!$. **10**

M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS (2004 Pattern)
MT-802 : Hydrodynamics (Old)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate marks.*

1. a) Explain Lagranges method of description and hence derive equation of continuity. **7**

   b) A two dimensional unsteady velocity field is given by $u = x(1 + 3t)$, $v = y$. Find the equation of stream line. **4**

   c) Derive the relation between potential function and stream function in polar co-ordinate system. **5**

2. a) Show that if the motion is irrotational, then the velocity vector is the gradient of a scalar function of position. **6**

   b) A two dimensional incompressible flow field has the x component of velocity given by the expression $u = e^{-x}(x \sin y - y \cos y)$. Determine y component of velocity. Is this flow irrotational ? **5**

   c) In a cylindrical co-ordinate system (r, θ, z) the radial component of velocity $\bar{q}(u, v)$ of a two dimensional flow is $u(r, \theta) = \frac{3}{2} r^{3/2} \cos\theta$. Find the expression for v when $v = 0$ at $\theta = 0$. **5**

3. a) State and prove Bernoulli's theorem for unsteady flow. **9**

   b) Test whether the motion specified by $\bar{q} = \dfrac{k^2(x\bar{j} - y\bar{i})}{x^2 + y^2}$ (k = constant) is of the potential kind and if so, determine the velocity potential. **7**

4. a) State and prove Kutta-Joukowski theorem. **8**

   b) State and prove the theorem of Blasius. **8**

5. a) Define vortex pair and find the complex potential of vortex pair. **8**

b) Find the equation of the stream lines due to uniform line sources of strength m through the points A(– c, 0), B(c, 0) and a uniform line sink of strength 2m through the origin. **8**

6. a) Define Stokes stream function. **5**

b) Discuss the flow due to a circular cylinder of mass m moving with velocity u. **6**

c) A two dimensional flow towards α normal boundary is found to be characterised by α normal component of velocity that varies directly with distance from the boundary. Determine the stream function. **5**

7. a) Explain shear rate, volumetric deformation and simple shear. **8**

b) The velocity components of a certain flow are given as $u = \propto (x + y)$, $v = b (x^2 – y^2) + 6y$, $w = – 2dz$ where a, b and d are constants. Represent the motion as the sum of rotation and deformation of fluid element. **8**

8. a) Obtain the relation between stress and rate of strain components. **8**

b) What is the complex potential for two-dimensional fluid motion ? Discuss the flow for which $w = z^2$. **8**

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[3721] – 405

M.A./M.Sc. (Semester – IV) Examination, 2010
(2008 Pattern)
MATHEMATICS
MT 805 : Lattice Theory (New)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Let the algebra $L = \langle L; \wedge, \vee \rangle$ be a lattice. Set $a \leq b$ if and only if $a \wedge b = a$. Then prove that $L^P = \langle L; \leq \rangle$ is a poset and the poset L^P is a lattice. **6**

   b) Let I be an ideal and let D be a dual ideal. If $I \cap D \neq \phi$ then show that $I \cap D$ is a convex sublattice, and every convex sublattice can be expressed in this form in one and only way. **6**

   c) Find all neutral elements of $C_2 \times C_3$, where C_i, i = 2, 3 are chains of i elements. **4**

2. a) Prove that I is a prime ideal of a lattice L if and only if there is a homomorphism ϕ of L onto C_2 with $I = \phi^{-1}\{0\}$. **6**

   b) Prove that if L is finite then L and Id(L) (ideal lattice of L) are isomorphic. **4**

   c) Let L be a lattice and Con (L) be the set of all its congruences. Then prove that Con (L) is a lattice. **6**

3. a) Prove that a lattice is modular if and only if it does not contain a pentagon. **8**

   b) State and prove Nachbin theorem. **8**

4. a) Let L be a distributive lattice with 0. Show that Id (L), the ideal lattice of a lattice L, is pseudo complemented. Is the converse true ? Justify. **8**

   b) State and prove Hashimoto theorem. **8**

P.T.O.

5. a) Let L be a finite distributive lattice. Then prove that the map $Q : a \to r(a)$, where $r(a) = \{j \in J(L) \mid j \leq a\}$, is an isomorphism between L and $H(J(L))$. **7**

b) Let L be a lattice, let P be a prime ideal of L, and let a, b, c $\in$ L. Prove that if $a \vee (b \wedge c) \in P$ then $(a \vee b) \wedge (a \vee c) \in P$. **5**

c) Prove that a lattice L is distributive if it satisfies : $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ for $x, y, z \in L$. **4**

6. a) Prove that every lattice is a chain if and only if its every ideal is a prime ideal. **5**

b) Prove that in a Boolean lattice, an ideal is maximal if and only if it is prime. **6**

c) Prove that any finite distributive lattice is pseudo complemented. **5**

7. a) State and prove Stone's separation theorem for a distributive lattice. **8**

b) Prove that in a modular lattice, an element is standard if and only if it is distributive. **6**

c) Show that $N_5 \cong L \times K$ implies that the lattice L or K has only one element. **2**

8. a) Prove that the set of all neutral elements of a lattice forms a sublattice. **6**

b) Prove that the complemented elements of a distributive lattice form a sublattice. **5**

c) Prove that every ideal of a distributive lattice is a standard ideal and conversely. **5**

[3721] – 405

M.A./M.Sc. (Semester – IV) Examination, 2010
(2004 Pattern)
MATHEMATICS
MT 805 : Field Theory (Old)

Time : 3 Hours | Max. Marks : 80

N.B.: 1) Attempt ***any five*** *questions.*
2) Figures to the ***right*** *indicate marks.*

1. a) Let k be a field and $F \subset E$ extension fields of k. Show that $[E : k] = [E : F]\,[F : k]$. **6**

   b) Let α be algebraic over a field k. Show that $k(\alpha) = k[\alpha]$. **5**

   c) Find the degree of $K = \mathbb{Q}\left(\sqrt{2}, i\right)$ over $\mathbb{Q}$. Justify your answer. **5**

2. a) Let $\alpha \in E$, where E is a field extension of a field F. Suppose L is a field containing F and let $\sigma : E \to L$ be an isomorphism over F from E into L. Let $f(x) \in F[x]$ be such that $f(\alpha) = 0$. Show that $\sigma(\alpha)$ is a root of f(x). **4**

   b) Let k be a filed and f a polynomial in k[X] of degree ≥ 1. Show that there exists an extension E of k in which f has a root. **5**

   c) Let K be a splitting field of the polynomial $f(X) \in k[X]$. If E is another splitting field of f, show that there is an isomorphism $\sigma : E \to K$ inducing identity on k. show also that if $k \subset k \subset k^a$, where k^a is an algebraic closure of k, then any embedding of E in k^a inducing the identity on k must be an isomorphism of E onto K. **7**

3. a) If K_1, K_2 are normal over k and are contained in some filed L, show that $K_1 \cap K_2$ is normal over k. **4**

   b) Let $E = F(\alpha)$, where α is algebraic over F, of odd degree. Show that $E = F(\alpha^2)$. **5**

   c) Let $E \supset F \supset k$ be a tower of fields. Show that $[E : k]_s = [E : F]_s\,[F : k]_s$ **7**

4. a) Construct a finite filed of 9 elements. **5**

b) Let E be a finite extension of a field k. Suppose there are only a finite number of fields F such that $k \subset F \subset E$. Show that there is $\alpha \in E$ such that $E = k(\alpha)$. **6**

c) Which of the following is a Galois extension ? Justify your answer.

i) $\mathbb{Q}\left(2^{1/3}\right) / \mathbb{Q}$ ii) $\mathbb{Q}(i) / \mathbb{Q}$ **5**

5. a) Let K be a field and let G be a finite group of automorphisms of K of order n. Let $k = K^G$ be the fixed field. Show that K is a finite Galois extension of k, and its Galois group is G. Show that [K : k] = n. **8**

b) Let $f(X) = X^3 - 3 \in Q[X]$. What is the splitting field of f(X) ? Find the Galois group of f(X), by explicitly writing all the automorphisms. **8**

6 a) Let K be a Galois extension of a field k with cyclic Galois group having 6 elements. Determine the number of intermediate fields between k and K. **5**

b) Let $f(X) = X^3 + aX + b \in Q[X]$ be an irreducible polynomial. What is the discriminant of f(X) ? State when the Galois group f(X) is A_3 and S_3. **5**

c) Let E/k be a finite extension. Let $\alpha \in E$. Define the trace $Tr_{E/k}(\alpha)$. Show that if E is a finite separable extension of k, then $Tr : E \to k$ is a nonzero functional. **6**

7. a) Let k be a field, n an integer > 0, (n, p) = 1, if ch. k = p > 0. Assume that there is a primitive n-th root of unity in k. Let K/k be a cyclic extension of degree n. Prove that there exists $\alpha \in K$ such that $K = k(\alpha)$, and α satisfies $X^n - a = 0$ for some $a \in k$. **8**

b) Let E be a separable extension of k. Suppose E/k is a solvable extension. Show that E is solvable by radicals. **8**

8. a) If n is odd > 1, show that $\phi_{2n}(X) = \phi_n(-X)$, where $\phi_n(X) = \prod_{\zeta}(X - \zeta)$, where ζ varies over primitive n-th roots of 1. **6**

b) Find the Galois group of the following polynomials :

i) $X^3 - X + 1$ ii) $X^2 - 2$. **5**

c) Show that the order of a finite field is always a power of a prime. **5**

M.A./M.Sc. (Semester – II) Examination, 2010
(2004 Pattern and 2008 Pattern)
MATHEMATICS
MT-602 : Differential Geometry
(Old and New)

Time : 3 Hours Max. Marks : 80

Instructions : *i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S. Prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. **6**

b) Find the integral curve through $p = (1, 1)$ of the vector field $f(x_1, x_2) = (x_2, -x_1)$. **5**

c) Sketch the level sets $f^{-1}(c)$, for n = 0, 1, of each function at the heights indicated

i) $f(x_1, x_2 ..., x_{n+1}) = x_{n+1}$; $c = -1, 0, 1$

ii) $f(x_1, x_2, ..., x_{n+1}) = x_1 - x_2^2 - ... - x_{n+1}^2$; $c = 0, 1.$ **5**

2. a) Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$, and let X be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some $t_0 \in I$, then prove that $\alpha(t) \in S$ for all $t \in I$. **6**

b) For $0 \neq (a_1, a_2, ..., a_{n+1}) \in \mathbb{R}^{n+1}$ and $b \in \mathbb{R}$, show that the n-plane $a_1x_1 + a_2x_2 + ... + a_{n+1}x_{n+1} = b$ is an n-surface. **5**

c) Find the length of the parametrized curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ defined by $\alpha(t) = \left(\sqrt{2}\cos 2t, \sin 2t, \sin 2t\right)$. **5**

3. a) The 1-sheeted hyperboloid H is defined as

$$-\frac{x_1^2}{a^2} + x_2^2 + ... + x_{n+1}^2 = 1 (a > 0).$$

What happens to the spherical image of H when $a \to \infty$? When $a \to 0$? **6**

b) Let $\{e_1, e_2\}$ be a pair of orthogonal unit vectors in $\mathbb{R}^3$, and $a \in \mathbb{R}$. Prove that $\alpha(t) = (\cos at) e_1 + (\sin at) e_2$ is a geodesic in the 2-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$. **4**

c) Find the curvature k of the oriented plane curve $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \ a \neq 0, \ b \neq 0.$ **6**

4. a) Let S be a 2-surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a geodesic in S with $\alpha \neq 0$. Prove that a vector field X tangent to S along α is parallel along α if and only if both $\| X \|$ and the angle between X and α are constant along α. **6**

b) Compute the Weingarten map for the circular cylinder $x_2^2 + x_2^3 = a^2$ in $\mathbb{R}^3 (a \neq 0)$. **6**

c) Define :

i) Gauss-Kronecker curvature

ii) Mean curvature. **4**

5. a) Let S be an n-surface in $\mathbb{R}^{n+1}$,oriented by the unit normal vector field N. Let $p \in S$ and $v \in S_p$. For every parametrized curve $\alpha : I \to S,$ with $\dot{\alpha}(t_0) = v$ for some $t_0 \in I$ prove that $\ddot{\alpha}(t_0) \cdot N(p) = L_p(v) \cdot v$. **6**

b) Find the normal curvature $k(v)$ for each tangent direction v at the given point p = (1, 0, ..., 0) of the given n-surface $x_1 + x_2 + ... + x_{n+1} = 1$ oriented by $\frac{\nabla f}{\|\nabla f\|}$. **6**

c) Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $f : S \to \mathbb{R}^k$. If f is smooth then prove that $f \circ \phi : U \to \mathbb{R}^k$ is smooth for each local parametrization $\phi : U \to S$. **4**

6. a) Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Prove that there exists an orthonormal basis for V consisting of eigenvectors of L. **6**

b) Let a > b > 0 and define $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\varphi(\theta, \phi) = \left((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi\right).$$

Show that φ is a parametrized 2-surface in $\mathbb{R}^3$. **6**

c) For each a, b, c, d $\in \mathbb{R}$, prove that the parametrized curve

$$\alpha(t) = \left(\cos(at + b), \sin(at + b), ct + d\right)$$

is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$. **4**

7. a) Let $\phi : U \to \mathbb{R}^{n+1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$ and let $p \in U$. Prove that there exists an open set $U_1 \subset U$ about p such that $\phi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$. **10**

b) Sketch the level set $f^{-1}(0)$ and typical values $\nabla f(p)$ of the vector field for $p \in f^{-1}(0)$, when $f(x_1, x_2) = x_1^2 + x_2^2 - 1$. **6**

8. a) Find the Gaussian curvature of the parametrized 2-surface

$$\varphi(\theta, \phi) = \left((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi\right) \text{ in } \mathbb{R}^3.$$ **8**

b) Let U be an open set in $\mathbb{R}^{n+1}$, let $f : U \to \mathbb{R}$ be a smooth function, and let $\alpha : I \to U$ be an integral curve of ∇f. Show that

$$\frac{d}{dt}(f \circ \alpha)(t) = \left\|\nabla f(\alpha(t))\right\|^2, \ \forall\, t \in I.$$ **5**

c) Sketch the surface of revolution obtained by rotating C about the x_1 axis, where C is the curve $x_2 = 1$. **3**

[3721] – 104

M.A./M.Sc. (Semester – I) (2008 Pattern) Examination, 2010
MATHEMATICS
MT-504: Number Theory

Time : 3 Hours Max. Marks : 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) If g is the greatest common divisor of b and c, then prove that there exist integers x_0 and y_0 such that $g = (b,c) = bx_0 + cy_0$. **6**

   b) Prove that if x and y are odd then $x^2 + y^2$ is even, but not divisible by 4. **5**

   c) Show that n^4+n^2+1 is composite if $n > 1$. **5**

2. a) Prove that if $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$. **6**

   b) What is the last digit in the ordinary decimal representation of 3^{400}? **5**

   c) Show that 2,4,6,....,2 m is a complete residue system modulo m if m is odd. **5**

3. a) Let p denote a prime. Prove that $x^2 \equiv -1 \pmod{p}$ has solutions if and only if $p=2$ or $p \equiv 1 \pmod{4}$. **8**

   b) Find all integers that give the remainders 1,2,3 when divided by 3,4,5 respectively. **4**

   c) Find all integers x and y such that $147x + 258y = 369$. **4**

4. a) Prove that for every positive integer n, $\sum_{d|n} \phi(d) = n$. **6**

   b) Find the highest power of 70 that divides 533 ! **4**

   c) i) Prove that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ if n is a positive integer.

   ii) Evaluate $\sum_{j=1}^{\infty} \mu(j!)$ **6**

P.T.O

5. a) Prove that, if p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left\{(p-1)/2\right\}\left\{(q-1)/2\right\}}.$$ **6**

b) Find the value of $\left(\frac{a}{p}\right)$ in each of the 12 cases, $a = -1, 2, -2, 3$ and $p = 11, 13, 17$. **6**

c) Find the value of $\left(\frac{-42}{61}\right)$. **4**

6. a) Prove that the product of two primitive polynomials is primitive. **6**

b) Prove that among the rational numbers, the only ones that are algebraic integers are the integers $0, \pm 1, \pm 2, \ldots$ (i.e.$\mathbb{Z}$). **5**

c) Find the minimal polynomial of the algebraic number $\frac{(1+\sqrt[3]{7})}{2}$. **5**

7. a) Prove that if α is any algebraic number, then there is a rational integer b such that $b\alpha$ is an algebraic integer. **6**

b) For any algebraic number α, define m as the smallest positive rational integer such that $m\alpha$ is an algebraic integer. Prove that if $b\alpha$ is an algebraic integer, where b is a rational integer, then m|b. **6**

c) Prove that $\sqrt{3}-1$ and $\sqrt{3}+1$ are associates in $Q\left(\sqrt{3}\right)$. **4**

8. a) Let m be a negative square-free rational integer. Prove that the field $Q\left(\sqrt{m}\right)$ has units ± 1, and these are the only units except in the cases m= –1 and m = –3. Prove that if m = –1 then units are ± 1 and $\pm i$ where as if m= –3 then units are

$$\pm 1, \frac{(1 \pm \sqrt{-3})}{2} \text{ and } \frac{(-1 \pm \sqrt{-3})}{2}.$$ **8**

b) If α and $\beta \neq 0$ are integers in $Q\left(\sqrt{m}\right)$, and if $\alpha \mid \beta$, Prove that $\overline{\alpha} \mid \overline{\beta}$ and $N(\alpha) \mid N(\beta)$. **5**

c) Prove that 1 + i is a prime in Q (i) **3**

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[3721] – 303

M.A.M.Sc. (Semester – III) (2008 Pattern) Examination, 2010
MATHEMATICS
(Optional) MT-703: Mechanics (New)

Time : 3 Hours Max. Marks : 80

*N.B.: i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) If the forces acting on a particle are conservative, show that the total energy is conserved. **5**

   b) Use D'Alembert's principle to determine the equation of motion of a simple pendulum. **5**

   c) A particle of mass m moves in xy plane with position vector $\overline{\Sigma} = i\, a \cos wt + j\, b \sin wt$, where a, b and w are positive constants and $a > b$. Show that

   i) Particle moves in ellipse

   ii) The force acting on the particle is always directed towards the origin.

   iii) The force field is conservative. **6**

2. a) Classify constraints with suitable examples. **5**

   b) Derive Lagrange's equation of motion from Hamilton's principle. **5**

   c) A particle of mass m moves in a plane under the action of a conservative force f with components.

   $F_x = -k^2\,(2x + y),\ F_y = -k^2(x + 2y),$

   where k is a constant. Find the total energy of the motion, the Lagrangian and the equation of motion of the particle. **6**

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3. a) Find the Euler-Lagrange differential equation satisfied by twice differentiable function y(x) which extremizes the functional

$$I(y\ (x)) = \int_{x_1}^{x_2} f\ (x, y, y^1)\ dx,$$

where y is prescribed at the end points. **6**

b) If L is a Lagrangian for a system of n degree of freedom satisfying the Lagrangian equations, then show that

$$L^1 = L + \frac{df(q_j t)}{dt}\ , j = 1.2, ... n,$$

also satisfies the Lagrangian equation, where f is any arbitrary, but differential function of its arguments. **5**

c) Show that the curve is a catenary for which the area of surface of revolution is minimum when revolved about y -axis. **5**

4. a) Reduce the two body problem to one body problem in central force motion of two bodies about their centre of mass. **6**

b) Derive the viral theorem, if the forces are derivable from a potential and show that $\overline{T} = \frac{n+1}{2}\overline{V}$. **5**

c) Find the shape of the plane curve of fixed length *l* whose end points lie on the x-axis and area enclosed by it and the x-axis is maximum. **5**

5. a) Define orthonormal transformation. Show that finite rotation of a rigid body about a fixed point of the body is not commutative. **5**

b) Define Eulerian angles. Find the matrix of transformation from a space set of axes to body set of axes interms of Eulerian angles. **6**

c) Obtain the Euler's equations for motion of a rigid body when one point of the body remains fixed. **5**

6. a) Derive Hamilton's principle for non-conservative system from D'Alembert's principle and hence deduce from it the Hamilton's principle for conesrvative system. **6**

b) Deduce Newton's second law of motion from Hamilton's principle. **5**

c) A particle of mass m is moving on the surface of the sphere of radius r in the gravitational field. Use Hamilton's principle to show the equation of motion is given by

$$\ddot{\theta} - \frac{p_{\phi}\cos\theta}{m^2 r^4 \sin^3\theta} + \frac{g}{r}\sin\theta = 0,$$

where p_{ϕ} is the constant of angular momentum. **5**

7. a) Define Posson's bracket and show that it is invariant under canonical transformation. **6**

b) If A is the matrix of a rotation through 180° about any axis. Show that if

$P_{\pm} = \frac{1}{2}(1 \pm A), P_{\pm}^2$ then $= P_{\pm}$. Obtain the elements of $P_{\pm}$ in any system. **6**

c) Derive with usual notation **4**

$$\frac{d}{dt}[u, v] = \left[\frac{du}{dt}, v\right] + \left[u, \frac{dv}{dt}\right]$$

8. a) Define and explain the following terms :

i) Degree of freedom

ii) Generalized momentum

iii) Virtual work **6**

b) Find the kinetic energy of rotation of a rigid body with respect to the principal axes in terms of Eulerian angles. **5**

c) For a particle the kinetic energy and potential energy is given by

$$T = \frac{1}{2} m\dot{r}^2$$

$$V = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{C^2}\right)$$

Find the Hamiltonian H and determine

1) Whether H = T + V

2) Whether $\frac{dH}{Dt} = 0$ **5**

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[3721] – 303

M.A/M.Sc. (Semester – III) (2004 Pattern) Examination, 2010
MATHEMATICS
MT-703: Functional Analysis (Old)

Time : 3 Hours Max. Marks : 80

Instructions : *i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) i) Define normed linear space.

ii) In normed linear space show that

A) $\big|\|x\| - \|y\|\big| \leq \|x-y\|$;

B) addition and scalar multiplication are jointly continuous on N. **8**

b) Give one example of Banach space with explanation. Is $\mathbb{R}^n$, a Banach space with the norm defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} ?$$

Justify your steps. **8**

2. a) Let M be a closed linear subspace of a normed linear space N, and let x_0 be a vector not in M, then prove that there exists a functional f_0 in N* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. **6**

b) Let N and N' be normed linear spaces and T a linear transformation of N into N'. Prove that the following conditions on T are all equivalent to one another :

i) T is continuous;

ii) T is continuous at the origin;

iii) T is bounded on N;

iv) If S is the closed unit sphere in N, then its image T(S) is a bounded set in N'. **8**

c) True/ False ? Justify your answer. **2**

If N is complete, then N is reflexive.

3. a) State and prove the uniform boundedness theorem. **6**

b) If N is a normed linear space, then prove that N is naturally imbedded into N **. **8**

c) If N is a Banach space, then prove that S = {x| ||x|| = 1} is complete. **2**

4. a) Define Hilbert space and give one example of Hilbert space with explanation. **6**

b) State and prove the parallelogram law. **4**

c) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **6**

5. a) If N is a normal operator on a Hilbert space H, then prove that $\|N^2\| = \|N\|^2$. **4**

b) Let H be a Hilbert space, and let $\{e_i\}$ be an orthonormal set in H. Prove that the following conditions are all equivalent to one another :

i) $\{e_i\}$ is complete

ii) $x \perp \{e_i\} \Rightarrow x = 0$

iii) If x is an arbitrary vector in H, then $x = \sum (x, e_i) e_i$

iv) If x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$. **8**

c) Show that the difference $P = P_1 - P_2$ of two projections on a Hilbert H is a projection on H if and only if $P_1 \leq P_2$. **4**

6. a) Prove that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself. **6**

b) If A is a positive operator on a Hilbert space H, then prove that I+ A is non singular. **6**

c) Prove that the adjoint operation $T \to T^*$ on B(H) has the following properties : **4**

i) $T^{**} = T$

ii) $\|T^*\| = \|T\|$.

7. a) With usual notations prove that $(l_1^n)^* = l_\infty^n$. **6**

b) Consider the operator T defined on l_2 by

$$T(x_1, x_2, x_3, x_4 \ldots.) = (0, x_1, x_2, x_3, x_4 \ldots.) .$$

Is T unitary ? Why ? **4**

c) Let y be a fixed vector in a Hilbert space H, and consider the function f_y defined on H by $f_y(x) = (x, y)$. Prove that f_y is a linear transformation, and $\|f_y\| = \|y\|$. **6**

8. a) If T is a normal operator on a Hilbert space H, then prove that M_i' s are pairwise orthogonal. **4**

b) If T is a normal operator on a Hilbert space H, then prove that each M_i reduces T. **4**

c) Let T be an operator on H, and prove the following statements :

i) T is singular if and only if $0 \in \sigma\{T\}$;

ii) If A is non singular, then $\sigma(ATA^{-1}) = \sigma(T)$. **8**

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