

SOME REFLECTIONS ON UNIVERSE AND NULL CLASSES

This paper attempts to examine some logical properties of the Universe and null classes in relation to one another from the point of view of extension instead of intension.

The Logic of algebra is generally characterized as the logic of classes. Classes are interpreted in two different ways : (i) in terms of intension and (ii) in terms of extension. The intension of a class is called its meaning and the extension is called its range of application or exemplification. Both extension and intension are, thus, regarded as two different aspects of the class. The laws of intension and extension do not apply to the same class in the same way since both intension and extension do not always go together. They vary with regard to certain classes. It is quite possible that two classes which have identical extension may not have identical intension. Take, for instance, the classes 'unicorn' and 'centaur'. These two classes have identical extension since they have no members or exemplifications and all classes that have no members or exemplifications are, by definition, identical. Such classes are called null classes. But in respect of intension null classes differ from each other. The concept of unicorn (or its meaning) is not identical with the concept of centaur (or its meaning). In other words, the meaning of one is different from the meaning of another.

When classes are interpreted in extension, it is said, they exhibit different relations to one another which their mere intensional interpretations do not exhibit. Relations which any two

classes exhibit to one another are usually characterized as class inclusions or class membership¹. When it is said that one class is included in another class, that simply means that the members of one class are also the members of another class.

The universe class is the class of all members and the null class is the class with no members. Both these classes, when interpreted in extensional sense, exhibit different relations to one another which their intensional interpretations do not reveal. Relations which they exhibit to one another are the following :

- i) The universe class, 1 and the null class, 0 are each other's complements.

This follows from the very definitions. The universe class is equal to the negation of the null class ($1 = \neg 0$) and the null class is equal to the negation of the universe class ($0 = \neg 1$). Since one class is the negation of another, any one of them can be taken as primitive and the other can be introduced by definition.

- ii) The universe class and the null class divide the universe of discourse exhaustively.

Since the universe class and the null class are each other's complements and all complementary classes divide the universe of discourse between them exhaustively, this becomes a logical truth.

- iii) The universe class and the null class are mutually exclusive.

The universe class and the null class are mutually exclusive since they have no members in common. What is included in one class is excluded from another class. This is consistent with their definitions.

- iv) The null class is included in every class.

From this it follows that the null class is included in the universe class. ($0 < 1$).

- v) The universe class includes every class

From this it follows that the universe class includes the null class in it. To say that the universe class includes the null class is to say the same thing that the null class is included in the universe class. This shows that both (iv) and (v) laws are mutually consistent. There is nothing in one which debars another from being true.

- vi) The product of the universe class and the null class is null ($1 \times 0 = 0$).

This follows since both classes are each other's complements.

- vii) The sum of the universe class and the null class is the universe class ($1 + 0 = 1$).

- viii) The universe class and the null class are not mutually inclusive or identical ($1 \neq 0$).

This follows from the (i) and (iii) laws.

- ix) The null class is the class of all incompatible properties, that is, it is the product (or conjunction) of two complementary classes ($A \times -A = 0$).

- x) The sum of two complementary classes is equal to the universe class ($A + -A = 1$).

From this it follows that the universe class includes complementary classes as its sub-classes. This also is obvious from the (vii) law.

- xi) The universe and null classes include themselves ($1 \subset 1$, $0 \subset 0$)².

The above mentioned laws exhibit different logical relations between the classes of universe and null in relation to one another. It is worth noting that all these laws are not fundamental laws. Some of them are fundamental laws and some of them are derived laws. Since for the present purpose this distinction is irrelevant, I shall not pursue it further.

Now let us examine the listed laws in relation to one another in order to get a better understanding about the classes of universe and null. Take the laws from (i) to (iii). These laws are perfectly consistent with one another. There is nothing in one which prevents the other from being valid. They jointly assert that the universe class and the null class are not only each other's complements but also divide the universe of discourse³ exclusively and exhaustively. The advantage of these laws is that they allow us to turn positive statements into negative and negative statements into positive. They function, thus, like quantifier negation rules. This is possible because complementary classes not only divide the universe of discourse between them but also are exclusive and exhaustive. But when we take these laws, that is, (i), (ii) and (iii) in relation to the laws of (iv) and (v), we find that they generate logical problem. Because, if we hold the validity of the laws from (i) to (iii) that is, the universe and null classes are not only each other's complements but also divide K between them exclusively and exhaustively then thereby we are compelled to maintain the view that the null class is neither included in the universe class nor the universe class includes it. But to maintain this view leads us to reject the validity of the (iv) and (v) laws since what they jointly assert is negated by the laws from (i) to (iii), that is, the null class

is neither included in the universe class nor the universe class includes it. And, suppose, if we hold the validity of the laws (iv) and (v) then again on the same ground we are led to reject the validity of the laws of (i), (ii) and (iii), that is, the universe and null classes are neither each other's complements nor do they divide K between them exclusively and exhaustively. Besides this, we also are led to reject their defining forms (i.e., $1 = -0$, $0 = -1$). From the above discussion, thus, it is clear that the laws from (i) to (v) are not mutually compatible laws since the acceptance of the laws from (i) to (iii) implies the rejection of the (iv) and (v) laws: and also conversely.

There is another difficulty. If the laws from (i) to (iii) are held to be valid then thereby it follows that the sum of two complementary classes in the universe class is a complementary class of the null class since the sum of two complementary classes in the universe class forms the universe class exclusively and exhaustively (i.e., $A + -A = 1$). This follows from the very definition of the term "complement". If it is so, none of the sub-classes of the universe class includes the null class. In other words, the null class is the class which is not included in any other class, since all non-empty classes are included in the universe class. But to accept this thesis means to reject the validity of the (iv) and (v) laws since they assert that the null is included in the universe class and the universe class includes it. The same thesis can also be established on this ground too: Every class has its complement. The greatest class is said to be the universe class. Its complementary class is the null class. Within the universe class two classes may be each other's complements (i.e., $A + -A = 1$). But the sum of the complementary classes in the universe class is not equal to that universe of discourse of which both universe and null classes are elements, since they differ in regard to their universe of discourse. The former's uni-

verse of discourse is the universe class of which A and $\neg A$ are two elements whereas the latter's universe of discourse is not itself a class; but instead is the entire domain, K of which 1 and 0 are two elements. Further, in the former case the universe class, I coincides with K whereas in the latter case the universe class is not K , but is one of the elements in K . Now, if A and $\neg A$ together form the universe class exclusively and exhaustively, and the universe class is a complementary class of the null class, then by the definition of word "complement" it follows that the null class is neither included in A nor $\neg A$ as it is asserted by the (iv) and (v) laws. And if we accept the validity of the (iv) and (v) laws then there by we are compelled to maintain the view that the laws (i) and (iii) are invalid since what is asserted by them is negated by the former laws; and also conversely. In other words, the acceptance of the validity of the former laws debar the latter laws from their being valid which proves that they are incompatible laws in relation to one another.

Let us consider the laws (iv) and (v) in relation to the laws (vi) and (ix). The (iv) and (v) laws jointly assert that the null class is included in the universe class and the universe class includes the null class. What they do not assert is this that the universe class is included in the null class or the null class includes the universe class. So far the laws (vi) and (ix) are concerned, they assert one and the same thing, that is, the null class is the class of incompatible properties. It is the class of things which, for instance, are both cats and not cats or everything and nothing. When these laws are taken separately they are perfectly alright. They do not generate difficulty. But the moment we relate them with each other they generate difficulty. For example, if we accept that the null class is the class of all incompatible properties along with the law that it is included in every class then it implies that all classes are the classes

of incompatible properties. If it is so, no propositions which express relations among classes will ever be true. Rather they will always be false since they assert incompatible properties and all propositions asserting incompatible properties are always false according to the law of contradiction.

But it must be noted that the above argument holds good only in those cases where the null class is the product (or conjunction) of complementary classes. It fails in those cases where the null class happens not to be the class of incompatible properties. A class owes incompatible properties only in relation to classes, for instance, as it is the case with the product null class, O of two complementary classes A and $\neg A$ respectively. From the above discussion, thus, it is evident that when we say a class is a null class, this does not necessarily mean that it is a class of incompatible properties since it is quite possible for a class to be null class without standing for incompatible properties. Of course, it fails provided that we take 'incompatibility' as a defining characteristic of 'null' class. Since as a matter of fact 'incompatibility' is not taken as a defining characteristic (or property) of 'null' class, the property of incompatibility does not follow from a class being a null class. What follows from a class being treated as a null class is that it has no members; it is a class of 'nothing'. This happens because 'null' class is defined in terms of a 'class of no members'. But if we take incompatibility as a defining property of null class along with the law that it is included in every class then thereby we are led to commit that no propositions which express relations among classes can ever be true. And, if we maintain the thesis in the restricted sense, that is, only the product null class of two complementary classes has the properties of incompatibility then we are not compelled to maintain the view that "No propositions which express rela-

ons among classes can ever be true" since it holds good only in those cases where the null class is the product of two complementary classes. It must be guarded that such propositions are always false not on any other ground except that they assert the conjunction of incompatible properties which fails to hold good in those cases where the null classes do not possess the properties of incompatibility.

Again, there is another difficulty. A proposition whose subject term happens to be null class, cannot, in the strict sense, be characterized as either true or false since the subject term of the proposition being a null class contains no members; and as a result the predication fails, that is in other words, neither affirmation nor negation of any property is admissible. Consider, for example, the sentences "The unicorns are animals" and "The unicorns are not animals". In both sentences the subject term stands for null class, that is, the class of unicorns. Since the class of unicorns is such a class which has no members in it, therefore, we cannot predicate the property of 'being animals' or 'not being animals' to them from the logical point of view. No predication of any sort is admissible. We can neither say that the unicorns are animals nor can we say that the unicorns are not animals. Nothing can be asserted. In regard to the unicorns the question of their being animals or not being animals does not arise at all since there are no unicorns. Since the sentences in which the term 'unicorns' occupies the position of the subject do not assert anything, positively or negatively, they do not express genuine propositions at all and consequently cannot be characterized as either true or false. This difficulty does not arise in those cases where the subject term happens to be non-empty term. In fact, the question of truth and falsity can legitimately be raised in the cases of those sentences whose subject term stands for null class only

when we grant the ontological existence of the non-existent entities like unicorns. But to maintain this view goes against the defining forms of the universe and null classes, that is, $1 = -0$ and $0 = -1$. We can neither introduce one class by negating another class nor can we turn positive statements into negative or negative statements into positive. Hence, instead of admitting the existence of non-existent entities, it is better to admit that propositions whose subject term stands for null class are not genuine propositions. They cannot be characterized as either true or false. They transcend the categories of truth values.

Now let us consider the (vii) law in relation to the laws (i), (ii) and (iii). The (vii) law asserts that the sum of the universe class and the null class is equal to the universe class ($1+0=1$). The operator sum, $+$ used here can have two possible interpretations: inclusive and exclusive. If we interpret the operator ' $+$ ' in the inclusive sense of the word 'or' then what the (vii) law asserts does hold good. But the acceptance of this kind of interpretation compels us to reject the validity of the (i), (ii) and (iii) laws since what is asserted by the (i), (ii) and (iii) laws is negated by the (vii) law, provided we interpret the inserted operator ' $+$ ' between 1 and 0 in the inclusive sense. Not only this, inclusive interpretation also compels us to reject the defining forms of the universe and null classes, that is, $1 = -0$, $0 = -1$. We can neither define 1 by negating 0 nor 0 by negating 1. Its disadvantage would be that we cannot turn positive propositions into negative or negative propositions into positive which the (i), (ii) and (iii) laws permit. And, suppose, if we go by the second interpretation, that is, if we interpret the symbol ' $+$ ' in the exclusive sense of the word 'or' then thereby we are compelled to reject the validity of the (vii) law that the sum of the universe class and the null class is the uni-

verse class. The exclusive interpretation of the symbol '+', inserted between 1 and 0 implies that their product class is a wider class than the class of universe, 1 of which both 1 and 0 are two different elements (It is worth noting here that logicians do not denote the sum of 1 and 0 as a separate class in the exclusive interpretation. It is rather called the universe of discourse, K of which 1 is an element). This follows on the ground that complementary classes form another class different from them (i.e., $A + \bar{A} = 1$) which is violated in the case of the universe and null classes (i.e. $1 + 0 = 1$). The sum of A and \bar{A} cannot be equal to 1 unless the operator + inserted between them is interpreted in the exclusive sense of the word 'or'. Not only this, both A and \bar{A} cannot be each other's complements unless they stand in exclusive relationship with one another. From the above discussion, thus, it is clear that the (i), (ii), (iii) and (vii) laws are not mutually compatible laws. We cannot maintain the thesis that the laws from (i) to (iii) are valid unless we reject the validity of the (vii) law in either of the interpretation of the symbol, '+'. It also holds good conversely.

A critical reflection on the (v) law reveals that the universe class is the greatest class among all the classes since it includes every class. If this is so, all classes other than the universe class are the sub-classes of the universe class. But to admit this view implies that the universe class is such a class which has no complementary class since no sub-classes of a class can legitimately be its complementary class and all classes other than the universe class are sub-classes of the universe class which rejects the validity of the laws from (i) to (iii). And, if we accept the validity of the laws from (i) to (iii), then thereby we are compelled not to accept the validity of the (v) and (vii) laws since they assert that the universe class includes the null class and

what is included cannot be complementary to that class in which it is included. From the above discussion, thus, it follows that the (i), (ii), (iii), (v) and (vii) laws are not mutually compatible laws. Because the acceptance of the laws from (i) to (iii) implies the rejection of the laws of (v) and (vii). It also is true vice-versa.

In short we can, thus, conclude that the properties of the universe and null classes, as exhibited by the laws from (i) to (x), are not compatible properties in relation to one another.

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NOTES

1. Logicians generally denote 'class inclusion' by the symbol ' \subset ' or ' \subseteq ' and 'Class membership' by the symbol ' \in '.
2. The symbol \subset has not been used here in the sense of 'less than'. It is rather in the sense of identity or total reflexivity.
3. It must be noted that the universe of discourse is different from the universe class. The universe of discourse is not by itself a class like the universe class. It rather stands for the entire domain (or context) of discourse under which both universe and null classes are formed. The universe of discourse, thus, is one of which both universe and null classes are elements. Logicians usually denote it by the letter K . The ground of their error is that we ignore the distinction between classifying individuals and relating classes to one another. In the domain K of individuals, the greatest class, I which can form out of these individuals happens to coincide with K and as a result it generates confusion. But when we relate classes to one another, the greatest class, I is not K ; but is an element in K like O .

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