Indian Philosophical Quarterly Vol.XIII Nos. 3 & 4, July-Sept and Oct-Dec. 86

SOME NEGLECTED PROBLEMS IN THE PHILOSOPHY OF MATHEMATICS

(1) THE FOUNDATIONALIST PHILOSOPHY OF MATHEMATICS :

I) Introduction : (a) Since the propositions of mathematics do not 'depend upon the actual concrete facts being just what they are,[1] that is why right from the antiquity mathematics has been subjected to philosophical investigations. Philosophers like Plato, Aristotle and Kant have developed their philosophies of mathematics which are known as classical philosophies of mathematics.

During the last quarter of the nineteenth century and the first quarter of the twentieth century, however, a distinguished type of philosophy of mathematics had a field day. This philosophy is often called the foundationalist philosophy of mathematics. There are three schools of this philosophy : the logicist, the formalist, and the intuitionist. The last one, i.e. intuitionist is not strictly a foundational study, neverthless it is regarded as a school within the foundationalists.

It is not our aim here to give a detailed account of either the classical or the foundational philosophies of mathematics. In fact, we shall not deal with the classical philosophies at all. We here provide a summary view of foundationalist concept of Philosophy of Mathematics and critically examine it to bring out its inadequacies. This would provide proper background in explaining our concept of Philosophy of Mathematics.

(b) The foundational studies emerged as a response to a critical situation that developed in mathematics during the nineteenth century. The three developements which led to this situation can be distinguished: the acceptance of non-Euclidean geometries as mathematical theories, the introduction of infinite sets as new mathematical entities, and re-organization of the calculus in terms of the number continuum as defined by Dedekind. Since the foundational studies were specifically directed towards a solution of the problems generated by these developments, these were bound to have certain limitations when considered as a philosophy of mathematics.

But it is surprising to find that the foundationalists have made virtue of their requirements and claimed that the answers they proposed to problems which arose at a particular stage of development of mathematics as a science are in fact the problems of philosophy of mathematics. We do not propose to state the specific problems that had arisen in the development of mathematics and the solutions suggested by the foundationalists, for that is not pertinent to out objective here. Suffice it is to note here that the suggested solutions were later proved to be inadequate.

What we wish to do here is to state and examine critically the conception of the philosophy of mathematics which Russell, a leader of the logicist school has formulated.

II) The Conception : (a) Bertrand Russell - (1875-1972) was one of the leading exponents of the foundationalists. According to him every subject matter 'can give rise to philosophical investigations as well as to the appropriate science, the difference between the two treatments being in the directions of movement and in the kind of truths which it is sought to establish.'[2]

Thus the philosophy of biology, for example, will begin with the subject matter of biology, namely the living organisms, just like the science of biology itself would begin. But the science of biology is interested in classifying the living beings first into plants and animals and then the animals into species etc. while the philosophy of biology will explicate the concept of a living being and by means of this explication seek 'to eliminate the peculiarity of the original subject matter, and to confine our attention entirely to the logical form of the facts concerned.'[3]

(b) Russell observes that philosophy and mathematics have a close affinity because both assert propositions which do not depend on concrete, actual facts. Therefore, philosophy and mathematics will be the same in all the possible worlds while special sciences like chemistry and biology will differ according to the facts of the worlds.

Coming to philosophy of mathematics proper Russell states that while mathematics begins with numbers and derives more and more complex propositions about numbers, the philosophy of mathematics seeks to go behind the concept of a number and to form the premisses of the science of arithmetic.[4]

As an achievement of his philosophy of mathematics Russell claims that it clarified the nature of objects of mathematics. They are neither physical nor mental, but they come from the world of logic.[5]

III) <u>The Criticism</u> : (a) The account of Russell's conception of the philosophy of science in general and that of mathematics in particular, clearly shows the inherent shortcomings in his philosophy of mathematics. These shortcomings broadly fall into two categories : the ambiguities in the formulation of its conception and the areas deliberately left out from the scope of this philosophy.

According to Russell the philosophy of a special science has not as its subject matter the "science itself", but the subject matter of that special science. Thus, for example, in the case of mathematics, the philosophy of that science will not concern itself with what mathematics does with numbers but will subject the very concept of a number to philosophical scrutiny. It is argued that this view, firstly, gives room to several ambiguities and secondly, that it allows too narrow a scope of philosophy excluding from it several areas of investigation which a philosophy should legitimately investigate.

Concenring the ambiguities we note the following :

(1) It is very difficult to pin-point the facts as starting points of inquiry with which a science begins. Russell, when he began formulating his philosophy of mathematics, noted that mathematics as a science began with numbers. Owing largely to the contributions made by Russell himself, we now hold that mathematics begins with sets. One can also maintain that mathematics begins with counting.

If we accept that the philosophy of a science is the analysis of the same facts with which that science begins we will not be able to distinguish between that science and its philosophy. This is what precisely happened in the case of Russell's philosophy of mathematics.[6] The most of the tools used by Frege and Russell such as the equivalence of two sets, the concept of an infinite set etc., were already developed in mathematics proper.[7] i.e., mathematics as a 'science'.

(3) It is not proper to specify a certain set of facts as the starting point of inquiry of a science. Only after a science is developed to a certain stage, one is in a position to say, that too temporarily that such and such a set of facts is fundamental to that science. Thus upto the nineteenth century, the natural numbers, the negative integers, the rational numbers and the irrational numbers were regarded as being of the same kind. It is only when the mathematicians reduced all the other categories of

numbers to the category of natural numbers did the natural numbers come to occupy the fundamental status in mathematics. Does Russell regard this reduction of other categories of numbers to the natural numbers a philosophical exercise or a mathematical (scientific) one? If so, on what grounds?

(4) Russell maintains that given a set of facts there are two directions of development : One is towards more complexity and the other is towards abstraction and simplicity. The former is the direction of the special science while the latter is that of its philosophy.[8] This is, to say the least, too simplistic a view of a science and its philosophy. This can be shown by considering the development of the theory of groups in mathematics. In one sense this is a development in the direction of abstraction and simplification, for a group in a structure with a single binary operation with certain properties which are obviously suggested by the properties of addition. In the other sense it is a development towards more complexity, for the theory was developed to solve the problem concerning the solution of an equation by radicals. The group theory, in this latter sense, is rightly regarded as an advanced theory in mathematics.

(b) On the other hand the areas of investigation which this conception of a philosophy excludes from its purview are also important and should constitute the field of investigation for any philosophy. We may classify these areas as pertaining to the general set of problems which are distinct from specific problems.

Russell's conception of philosophy of mathematics excludes from its analysis the mathematical theories themselves. Therefore his philosophy does not discuss the nature of the mathematical processes, the certainty or otherwise of mathematical results, the methodology and criteria of progress in mathematics, the demarcation of mathematics from other subjects such as physics or chemistry, the process of discovery of a new theory in mathematics and its justification in the shape of its being accepted as a bonafide mathematical theory and so on.

At the level of the inquiry concerning specific problems, mathematics totally ignores the following issues : the status of an individual working mathematician within a community of mathematicians, the nature of collaboration between two mathematicians, how a particular problem comes to be regarded as a mathematical problem, how the individual mathematician perceives it, from several alternative avenues of solving the given problem why a particular one is selected, the mode of presenting the solution of the problem and so many others.

Finally, the Russellian philosophy fails to locate the mathematical thinking in the fabric of culture. It does not discuss what is the relationship betweeen mathematics on the one hand and other facets of life on the other. Most importantly, it totally ignores the tremendous success mathematical processes have achieved in such disparate disciplines as Economics, Physics and Astronomy.

(2) TENETS OF A MODERN PHILOSOPHY :

I) The Mathematical Activity : (a) Russell regarded the subject matter of mathematics as the field of investigation for a philosophy of mathematics. We hold that it is the nature of mathematical theories and not what the theories are about should be the object of investigation for a philosophy of mathematics. Thus it is not numbers about which the philosophical discussion should centre but how mathematicians deal with numbers, what makes them classify the numbers into even and odd numbers or prime and composite should be the questions of interest for a philosophy of mathematics. When Frege defined a number as an equivalence class of sets he was contributing to mathematics and not to its philosophy.

Thus a philosophy of mathematics must start with an assumption that there is a corpus of theories which goes under the name of mathematics and not with a range of facts with which mathematics deals. Now the philosophy of mathematics comes into significance when the identified corpus of mathematical theories is saught to be augmented by adding some new theories; and this entry of new theories into mathematics is obstructed by a section of mathematical community.

At a given point of time how is a body of results to be demarcated as mathematics? The only ground about which no serious controversy can take place is the ground of tradition. Some result belongs to mathematics because traditionally it has been so regarded. Now the concept of a tradition is generally invoked in the context of a collaborative human activity. Therefore, a philosophy of mathematics must start from the premises that its object of investigation is the outcome of a distinguished and largely collaborative human activity which we call the mathematical activity.

(b) That the mathematical activity is a collaborative activity can be readily granted; for a mathematical result, almost in every case, depends upon several results proved by other mathematicians. But once we assume the collaborative nature of mathematical activity we have to accept the objectivity of mathematical results. Russell observes, "Some have argued that the objects of mathematics were obviously not subjective and therefore must be physical and empirical; others have argued that they were obviously not physical, and therefore must be subjective and mental ..... Frege has the merit of accepting both the denials and finding a third assertion by recognising the world of logic, which is neither mental nor physical."[9]

Russell's quotation given above points towards anoter problem, namely, the ontology of mathematics, which we shall discuss later. For the present it is sufficient to note that the logicists agreed that the mathematical results and the mathematical objects could be rationally i.e. impersonally discussed and analysed and therefore though they had no physical existence, they led an objective existence.

This observation opens up one avenue of investigation of the mathematical activity. A mathematical result is a mental construction of an individaul mathematician. But it has an objective existence in the sense that it can be discussed and analysed without reference to the author of the construction. If so, the mental processes of the author are themselves, to a large extent, capable of being investigated without the help of psychology and mostly from the examination of the objectively set problem to which the construction is offered as a solution. This type of investigation may be regarded as the logic of mathematical discovery and as a process of analysing the mathematical activity.

(c) The objectivity of the mathematical results and the collaborative character of the mathematical activity suggest that the mathematical activity is a controlled activity. The debate between the formalists and the intuitionists is instructive in this respect. According to the formalists the new construction must be consistent with the previous results.[10] The intuitionists hold that the new element be strictly constructible from the old ones.[11]

It is now well known that the formalists failed to establish the consistency of arithmetic, which is the most basic branch of mathematics. On the other hand the concept of constructibility as has been formulated by the intuitionists is so restrictive that according to that concept many of the valuable branches of mathematics will have to be rejected.

Therefore, the control on mathematical innovation has not been exercised, on historical evidence, by any ad hoc principle such as that of consistency of constructibility.

We find that at every stage an innovated object or a result was accepted on two grounds : the existing state of available mathematics allowed the construction of the new object and secondly, the mathematical tradition felt the need for the acceptance of the new object. This can be illustrated by taking the example of the noneuclidean geometries.[12] Thus the two influences which control mathematical innovation are the tradition and the state of existing mathematics.

II) The Mathematical Reality : (a) The objectivity of mathematical results is so overwhelming that even the author of a result is inclined to regard it more as recording of an observed fact than as his own innovation. Therefore, Hardy (1873-1945) insists that there is an external reality which is called mathematical reality and the analysis of its nature must be, perhaps, the only problem for philosophy of mathematics[13].

In the philosophy of mathematics there has been the age-old concept of platonic reality. This reality is timeless and immutable. On the other hand the intutionists' conception of a mental construction insists that before a mental construction is made, the constructed object was not in existence.[14] Our discussion of mathematical activity makes it clear that we regard mathematics as the fruit of a human activity and therefore, the platonic concept of a mathematical reality does not hold good for us.

Therefore we must regard mathematical reality as consisting of the objective thought contents of results, theories and concepts. This reality is governed by the mathematical tradition and enlivened by the mathematical intuition. This latest element, i.e., 'mathematical intuition' requires some explanation.

(b) By the mathematical intuition we mean a feel for mathematical objects such as concepts, theories and results. This feel is not sensory because the mathematical objects are not concrete entities eventhough they are taken to have an objective existence. The mathematical intuition makes the mathematician aware of the incongruities in the mathematical reality and enables him to innovate new elements which will dissemble and reconcile those incongruities. This intuition is internal to a person but can be analysed impersonally because the directions that the intuition suggests are not arbitrary or subjective.

An incongruity in the mathematical reality usually arises becuase the domain of truth of some proposition in mathematics is sought to be expanded. This extension is also suggested by the entry of certain new concepts into the mathematical reality. To reconcile the expanded domain with the original proposition is the objectively set problem for the mathematician.

Through exploring the available reality the mathematician gets hold of a new configuration of theories, propositions and concepts and presents it in the form of a discourse. This is precisely the character of a mathematical innovation. One can illustrate it by considering Lebesgue's theory of integration. Cauchy's proposition was that every function was integrable. Here the domain of truth was the collection of functions and at the time of Cauchy a function meant a continuous function. But later the concept of a function was extended to include functions which were discontinuous every where. Thus arose an incongruity. Lebesgue's (1875-1941) exploration of reality yielded two new concepts : the concept of a measurable function and that of. a measurable set. Thus the new proposition emerged : Every measurable function is integrable.[15]

The innovated object then stands as a candidate for admission into the mathematical reality. Such an admission is governed by the mathematical tradition.

(c) The criteria used by the tradition for the acceptance of innovation are generally known. The - innovated object stands independently of tradition because of its intrinsic value, but it is accepted by the tradition because of its instrumental value. Two criteria are used in this regard. They are : usefulness and genuineness. The genuineness in the sense that the new object must stand the tests of rigour and truth.

Thus the main elements of a modern philosophy of mathematics could be the following: the objective existence of mathematical concepts and results, the mathematical tradition, the mathematical intuition and the logic of mathematical discovery. All these elements have to be so arranged as to give a coherent view of mathematics which can also account for the relationship between mathematics and a larger canvas of life.

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NOTES

1. Our Knowledge of the External World P.190 Bertrand Russell, George Allen and Unwin Ltd. London 1972.
2. Ibid.
3. Ibid.
4. Ibid.
5. Ibid, page 206.

6. Proceedings of the Fifth Canadian Mathematical Congress 1961 Editor Rosenthall. University of Tarento Press, Torento 1965. P.198.

7. Contributions to the Theory of Transfinite Numbers by G. Cantor. Dover Publications Inc. New York (Year not given).

8. Russell op-cit. P.190

9. Russell op-cit. P.210

10. Philosophy of Mathematics. P.135 Putnam and Benacerraf (editors) Englewood Cliffs : Prentice Hall Inc. (1964).

11. Philosophy of Mathematics P.28 by Korner, Stephan. Hutchinson University Library, London 1971.

12. See Mathematics and Logic in History and Contemporary Thought. by Carrucio Etter. Translation by Quighy, Isabel. Faber and Faber. London (1964).

The eliptic non-euclidean geometry was actually discovered by Saccheri (1667-1733) but it has to wait for its acceptance as a mathematical theory till about the middle of the 19th century. This was because the accredited mathematical theories of 18th century could not countenance the strange results incorporated in the new geometry. By the middle of the 19th century the mathematical tradition cherished the view the the nagging problem posed by the fifth postulate of Euclid can only be solved in terms of non-Euclidean geometries.

13. A Mathematician's Apology, PP.123-24 by G. H. Hardy Cambridge at the University Press, Cambridge, 1967.

14. Intuitionism : An Introduction Heyting, A. North Halland Publishing Co. Amsterdam (1956). P.3

15. For Labesgue's theory of Integration see Toward a Theory of Mathematical Research Programme by Hallett Michael, BJPS Vol.30 (1979) and also Lebesgues Theory of Integration by Hawkinsk Thomos. University of Wisconsin Press, London (1970).