USE OF HYPOCYCLIC CURVES IN ART

It is said that mathematics is an art. The origin of mathematics is a physical reality. So is that of Art. The legendary Leonardo-da-Vinci used mathematics in his attempts to represent reality on canvas and thus even laid the foundation of a branch of mathematics called Projective Geometry.

Very often, mathematical curves have been used by artists. Beautiful art-forms are obtained from a given curve by successive variation of a constant called parameter of the curve, in a uniform manner. The uniform change in the parameter brings about an uniform change in the shape of the curve and creates some art-form.

One of the most interesting family of curves is that of the Hypocyclic Curves. These curves have very beautiful shapes and moreover the curve strongly depends on certain constants called parameters of the curve. A slight variation in a parmeter can bring an unexpected and aesthetic change in the shape of the curve.

In this article, we give an account of the change in the shape of hypocyclic curves as the parameters change. We have avoided many laborious calculations and many statements which are just stated can be justified by rigorous mathematical proof.

Eduction of the Curve

When a circle rolls inside a steady circle such that the rotating circle is always in contact with the steady circle, then the curve traced by any point within the rotating circle is called a hypocyclic curve.

In figure (1) let 'O' be the center of the steady circle and 'a' be its radius. Let 'A' be the centere of the rotaing circle and let 'b' be its radius. Let P be any point within the smaller circle. So that $AP = Rb$ ($o \leqslant R \leqslant 1$). We will find the equation of the curve traced by P.

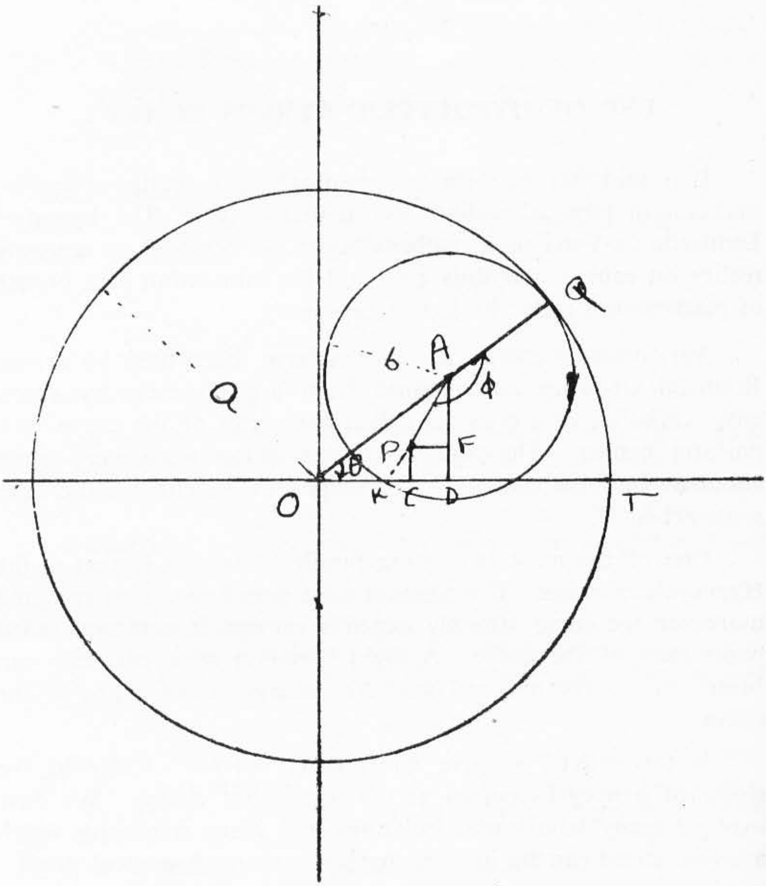


Fig. 1

Let Q be the point of contact of the two circles at any instant. Let $Q\hat{O}T = \theta$ $P\hat{A}Q = \phi$, Let ' K ' be a point on the circumference of the smaller circle so that P lies on the line AK. Let initially K and T coincide.

Then we have Arc KQ=Arc QT.

Hence $a\theta = b\phi \quad \therefore \phi = \frac{a}{b}\theta$

Let $PC \perp OT$ and $AD \perp OT$ and $PF \perp AD$

Let $P = (x(\theta), y(\theta))$

Then $x(\theta) = 0D - DC$ $y(\theta) = AD - AF$

Now $\phi + (90 - \theta) - \alpha = 180$ $\therefore \alpha = \phi - \theta - 90$

Hence $X(\theta) = OA \cos\theta + AP \cos[180 - (\phi - \theta)]$

$= (a - b)\cos\theta + AP\cos(\phi - \theta)$

$= (a-b)\cos\theta + Rb\cos\left(\frac{a}{b} - 1\right)\theta$

$y(\theta) = OA \operatorname{Sin}\theta - AP \operatorname{Sin}[180 - (\phi - \theta)]$

$= (a - b)\operatorname{Sin}\theta - Rb\operatorname{Sin}(\phi - \theta)$

$= (a - b)\operatorname{Sin}\theta - Rb\operatorname{Sin}\left(\frac{a}{b} - 1\right)\theta$

Hence the equations of the hypocyclic curves are :—

$x(\theta) = (a - b)\cos\theta + Rb\cos(a/b - 1)\theta$

$y(\theta) = (a - b)\operatorname{Sin}\theta - Rb\operatorname{Sin}(a/b - 1)\theta$

These are called the *freedom equations*, of the family of Hypocyclic curves. Since as the paramaters a, b, R change, the curve changes its shape.

The nature of the curve depends very much on the ratio a/b and interesting properties of these curves are derivable from this ratio.

Initially, we have assumed that K and T coincide. At this position the point P is at minimum distance from O.

The next maximum of OP will occour when P is nearest to Q i.e. after one complete rotation. And the sucessive maxima will also occur similarly.

The first maximum occurs when Q and T coincide i.e. when $\phi = \theta = 0$. The next maxima occur after complete rotations of the smaller circle, i.e. when $\phi = 2\pi$; $\phi = 4\pi$; $\phi = 6\pi$;....etc. the maxima will be formed. i.e. when $\phi = 2n\pi$ $n = 0, 1, 2, \ldots\ldots$ The curve attains maximum for OP.

Now since $\phi = (a/b)\theta$ Hence when $\theta = 2n\pi\, b/a$ the maxima are formed. If β maxima are there in the curve which is completed in γ revolutions of the smaller circle,

then $\theta_{max} = 2\pi\gamma$

hence $2\pi\gamma = 2\pi\beta\, b/a$. Hence $\gamma = b/a\, \beta$

i.e. $\beta/\gamma = a/b = \dfrac{\text{number of maxima of OP}}{\text{number of revolutions.}}$

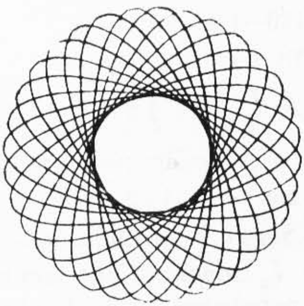
Similarly, minima of OP is attained at $\theta = (2n+1)\, b/a$ $n = 0, 1, 2 \ldots\ldots$

Obviously, there are β number of minima and again,

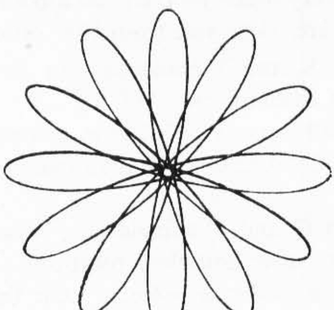
$$\frac{\text{number of minima of OP}}{\text{number of revolitions}} = a/b$$

Thus the curve fluctuates between maxima and minima the number of which is completely determined by the ratio a/b.

Hence by using circles with different ratio of radii we can have hundreds of types of these curves.



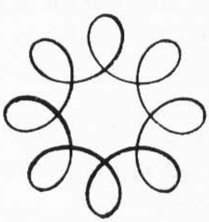
(Fig. 2 (*a*), 2 (*b*), 2 (*c*) show some of these curves)



In figure 2 (*a*), $\frac{a}{b} = \frac{32}{15}$

in 2 (*b*) $\frac{a}{b} = \frac{12}{7}$

in 2 (*c*) $\frac{a}{b} = \frac{8}{7}$



Moreover the number of maxima or minima of OP does not depend on R. i. e. the position of point — P.

For this reason a/b is called the characteristic ratio of the curve. It determines the nature of the curves which can be drawn by using two circles with given radius a, and b.

Now
$$\text{OP}(\max) = a - (1-R)\, b = G \max$$
$$\text{OP}(\min) = |\, a - (1+R)\, b \,| = G \min$$

Thus the curve lies within two circles with radius $a - (1-R)\, b$ and $|\, a - (1 + R)\, b \,|$.

As R decreases to zero then G max decreases to $(a - b)$ and G min increased to $(a - b)$.

Thus as R decreases the curve decreases in size and ultimately merges into a circle with radius $(a - b)$

Some special Hypocyclic Curves

When R=1 i.e. point P is on circumference of the rotating circle, then the curve is a Hypocycloid; when $a/b = 3/1$, we get a Deltoid

$$x(\theta) = 2\, b \cos\theta + b \cos 2\theta$$
$$y(\theta) = 2\, b \sin\theta - b \sin 2\theta$$

When $a/b = 4/1$ we get an Asteroid

$$x(\theta) = 3\, b \cos\theta + b \cos 3\theta$$
$$y(\theta) = 3\, b \sin\theta - b \sin 3\theta$$

When $a = 2\, b$

then
$$x(\theta) = b\,(1 + R) \cos\theta$$
$$y(\theta) = b\,(1 - R) \sin\theta$$

This is an equation of an ellipse with major axis $b\,(1 + R)$ and minor axis $b\,(1 - R)$ when $R = 1$ Then $y(\theta) = O$

Hence the hypocyclic curve is a straight line, when $\frac{a}{b} = \frac{2}{1}$ and $R = 1$.

(In the sequel maximum and minimum will allways refer to maximum and minimum of OP.)

Variation in the curve due to change in R

We will consider for simplicity two curves when (1) $\frac{a}{b} = \frac{3}{1}$ and (2) $\frac{a}{b} = \frac{3}{2}$. The results obtained for those curves can be generalised.

(1) $\frac{a}{b} = \frac{3}{1}$ Then $x(\theta) = 2b\cos\theta + Rb\cos 2\theta$

$$y(\theta) = 2b\sin\theta - Rb\sin 2\theta$$

(2) $\frac{a}{b} = \frac{3}{2}$ Then $x(\theta) = \frac{b}{2}\cos\theta + Rb\cos\frac{\theta}{2}$

$$y(\theta) = \frac{b}{2}\sin\theta - Rb\sin\frac{\theta}{2}.$$

Now when $R = 1$ both the curves, (1) and (2) are very much similar in appearance. They have 3 cusppoints (maximas).

But as R changes, both the curves evolve in a very different manner. Consider the curve (1)

$$x(\theta) = 2b\cos\theta + Rb\cos 2\theta$$
$$y(\theta) = 2b\sin\theta - Rb\sin 2\theta$$

The curvature ρ of this curve can be easily shown to be

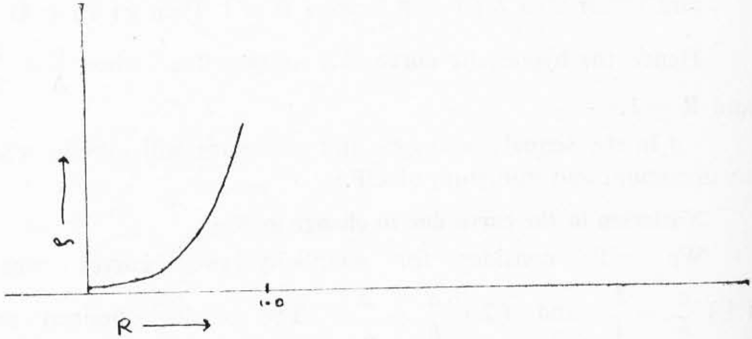
$$\rho = \frac{1}{2b}\,\frac{1-2R^2 + R\cos 3\theta}{(1+R^2-2R\cos 3\theta)^{3/2}}.$$

When $\theta = \frac{2}{3}\pi j$ ($j=0,1,2,\ldots\ldots$) i.e. at the maxima, we have

$$\rho = \frac{1}{2b}\,\frac{1-2R^2+2R}{(1+R^2-2R)^{3/2}} \quad \therefore\ \rho = \frac{1}{2b}\,\frac{(1+2R)}{(1-R)^2}$$

When $R = 1$ then $\rho = +\infty$ and ρ decreases,

as R decreases and ultimately $\rho = \frac{1}{2b}$ when $R=0$. (See graph 1)



Graph I—Change in the curvature at Maxima as R changes.

At minima i.e. when $\theta = (2\pi j + 1)\,\pi/3 \;\; (j = 0, 1, 2, \ldots.)$

We have $\rho = \frac{1}{2b} \; \frac{1 - 2R^2 - R}{(1 + R^2 + 2R)^{3/2}} = \varrho$

$$\varrho = -\frac{1}{2b} \; \frac{2R^2 + R - 1}{(1 + R^2 + 2R)^{3/2}}$$

Hence $\rho = 0$ when $2R^2 + R - 1 = 0$

i.e. when $\frac{R = -1 \neq \sqrt{1+8}}{4}$ i.e. $R = \frac{1}{2}$

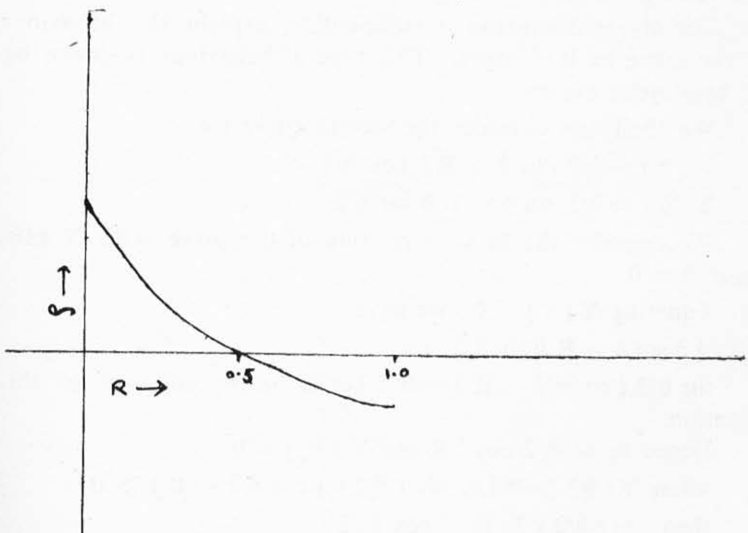
(Since $R = -9/4$ is not possible).

Hence when $R = \frac{1}{2}$ $\rho = 0$ at the point of minimum.

If $R > \frac{1}{2}$, Then $2R^2 + R - 1 > 0$ Hence $\rho < 0$.

The negative value of the curvature indicates that in this case the curve is concave outwards. If $R < \frac{1}{2}$ then $\rho > 0$ hence the curve is concave inwards (i.e. towards the origin) and finally when $R = 0$; $\rho = \frac{1}{2b}$

which is the maximum value of ρ at this point.



Graph II—Change in The cutvature at Minima as R changes.

When $R = 1$, ρ is negative for all values except for those values for which maxima is attained.

(Since $\rho = \frac{1}{2b} \; \frac{\cos 3\theta - 1}{(2 - 2\cos 3\theta)^{3/2}}$ and since

$\cos 3\theta - 1 < o$ for all θ Except $\theta = \frac{2\pi}{3} j$; $j = 0, 1, 2, \ldots$

we have $\rho < 0$).

Similary it can be shown that the curvature of the curve when $R = \frac{1}{2}$ decreases from $\frac{4}{b}$ to zero as we go from point of maximum to point of minimum along the curve.

Further it can be shown that when $R < \frac{1}{2}$; $\rho > 0$ always.

Hence it can be easily concluded and indeed, can be proved rigorously, that when $0 < R < \frac{1}{2}$ then as we go from maximum to minimum, curvature goes on decreasing and is zero somewhere between maximum and minimum and then becomes negative and as R decreases the portion of the curve with negative curvature goes on decreasing and it vanishes when $R = \frac{1}{2}$.

The above discussion is sufficient to explain the behaviour of the curve as R changes. This type of behaviour is shown by all hypocyclic curves.

We shall now consider the second curve i. e.

$X(\theta) = b/2 \cos\theta + R\, b \cos \theta/2$

$Y(\theta) = b/2 \sin\theta - R\, b \sin \theta/2$

We consider the first intersection of the curve with X axis, Near $\theta = 0$.

Equating $Y(\theta) = 0$ we have

$\frac{1}{2} b \sin\theta = R\, b \sin \theta/2$ i.e.

$\sin \theta/2 (\cos \theta/2 - R) = 0$. Let θ_0 be the solution of this equation.

Hence $\theta_0 = + 2 \cos^{-1} R$ and $Y(\theta_0) = 0$

when $Y(\theta) > 0$ i.e. $\sin(\theta/2)(\cos \theta/2 - R) > 0$

then $\cos(\theta/2) > R = \cos \theta_0/2$

now $\max(\theta_0/2) = \pi/2$. Hence $\theta_0/2 \leq \pi/2$ always.

Now if $\frac{\theta}{2} < \frac{\theta_0}{2}$ then $\cos \theta/2 > \cos \theta_0/2$

Hence $Y(\theta) > 0$. For every θ such that $0 < \theta < \theta_0$. Hence within the range ($0 < \theta < 2\cos^{-1} R$) $Y(\theta)$ is positive and decreases to zero.

Similarly when $\theta_0 = -2\cos^{-1} R$ i.e.

($0 > \theta > 2\cos^{-1} R$); $Y(\theta)$ is negative and increases to zero.

When $\theta_0 = +2\cos^{-1} R$ and $\theta_0 = -2\cos^{-1} R$,

we have $X(\theta) = \frac{1}{2} b \cos(2\cos^{-1} R) + b R^2$.

Hence in the region $-2\cos^{-1} R \leqslant \theta \leqslant 2\cos^{-1} R$

the curve forms a 'Loop' at $\theta = 0$ (i.e. the maximum length of the loop is along X axis.)

Simiarly curve forms loops, for other values of θ for which the maximum is attained.

Let $P = (x_0, y_0)$ be, (figure 3), the lower point of the loop and $Q = (x_1, y_1)$ be the point at which maxima is attained.

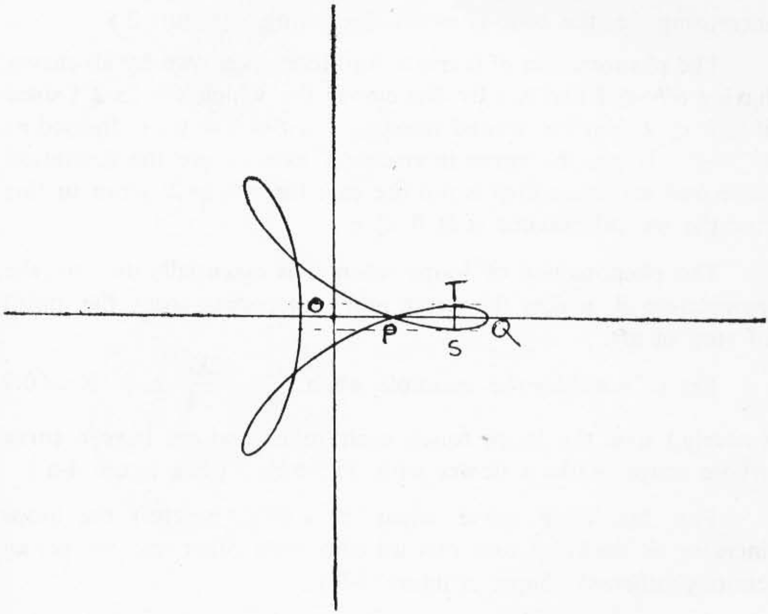


Fig. 3

Then at $\theta = 0$

$P = (b/2 \cos\{2\cos^{-1} R\} + b R^2, 0) = (x_0, y_0)$

$Q = (a + R b, 0) = (x_1, y_1)$

Now since x_0 decreases faster than x_1 when R decrease, PQ will increase as R decreases i.e. the loop will increase in length as R decreases.

Since PQ is independent of θ hence the length of loop will be same for all loops for given R.

When $R = \frac{1}{2}$ then $P = (0, 0)$

i.e. the loops are formed at the origin.

When $\frac{1}{2} < R \leqslant 1$ then $x_0 > 0$

When $0 \leqslant R < \frac{1}{2}$ then x_0 is negative hence the origin lies within the loop. (In fact within every loop).

When $R = 1$ then $P = Q = (a + b, 0)$ hence no loop is formed.

It can further be shown that when R decreases TS goes on increasing i.e. the loop goes on thickening. (Figure 3)

The phenomenon of formation of loops is shown by all curves having $a/b < 2$ and not by the curves for which $a/b \geqslant 2$ (since if $a/b < 2$ then the second maxima (first at $\theta = 0$) is formed at $\theta > \pi$. Hence the curve intersects X axis before the formation of second maxima which is not the case for $a/b \geqslant 2$, since in this case the second maxima is at $\theta < \pi$.

This phenomenon of loops when it is essentially due to the parametere R, makes the curve more interesting from the point of view of art.

Let us consider the example when $\frac{a}{b} = \frac{35}{24}$ and $R = 0.9$ (nearly) then the loops touch each other and we have a curve whose shape is like a flower with 35 petals. (See figure 4-a)

For the same curve when $R = 0.7$ (nearly) the loops increase in thickness and mix up with each other and we get an entirely different shape. (figure 4-b)

When $R = 0.4$ the loops are so much mixed with each other that it is not easy to distinguish them separately (figure 4-c).

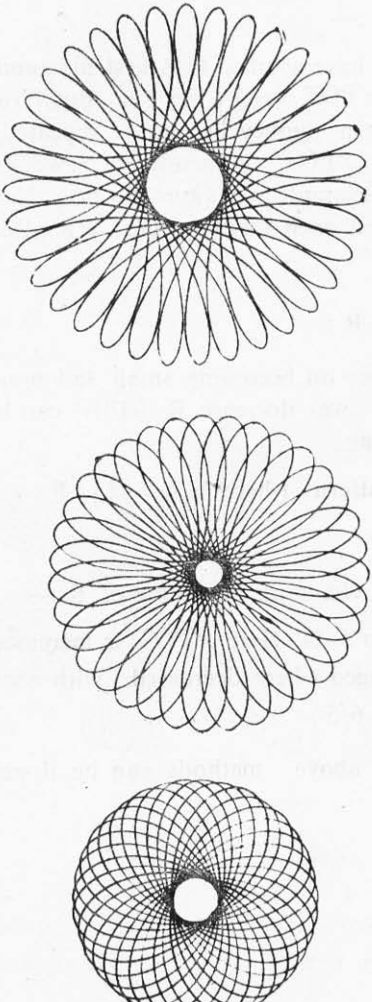


Fig. 4-(*a*), 4-(*b*), 4-(*c*)

Combination of the curves to form new patterns

Till now we have considered how the hypocyclic curve changes shape as the parameter changes. Now we will discuss some simple methods, to combine these hypocyclic curves.

(1) Rotation of the curve :—

In the discussion above we have asumed that first maximum is on X i.e. at first maximum $\angle POT = O$ (Fig I). But if we relax this condition and draw a hypocyclic curve repeatedly (a, b,r are fixed) by changing $\angle POT$ uniformly then we get very beautiful patterns. For example see figure 5 (b). Here a/b=7/4 and $\angle POT$ is increased each time by 3.75°. (R=0.75 nearly).

(2) Changing the value of R :—

We know that the curve goes on becoming small and more and more ' convex inwards ' as we decrease R. This can be utilised to draw beautiful patterns.

Figure 5 (a) shows a pattern [here $a/b = \frac{32}{25}$ R = .9 and R = .5

(3) Rotating the curve and changing the value of R :—

This method is illustrated in 5–D. Here $\angle POT$ is increased and at the same time R is changed (here decreased) with each incriment in ($\angle POT$ Here a/b=6/5)

Easy generaliation of the above methods can be done.

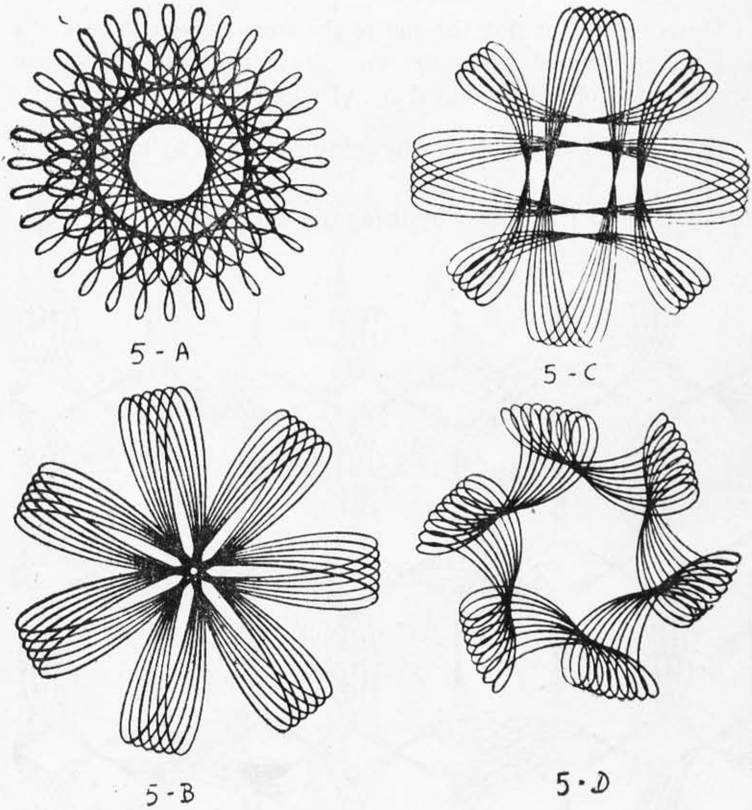


Fig. 5 (*a*), 5 (*b*), 5 (*c*), 5 (*d*).

Many interesting patterns can be obtained in this manner by rotating the curves in different manner.

We have discussed uptill now about the curves with same characteristic ratio. But we can combine curves with different characteristic ratio and obtain beautiful art forms. For example Figure 5-c. [Here two types of curves one with $\frac{a}{b} = \frac{2}{1}$ and other with $\frac{a}{b} = \frac{4}{3}$ are used].

Moreover this is not the end of the story. We can combine the patterns formed by using the above methods and obtain beautiful pieces of Mathematical art. (Two of them are given here)

Eg. Figure 6 (a) is drawn by using the curve with $\frac{a}{b} = 5/6$ and Figure 6 (b) is drawn by using the curve with $\frac{a}{b} = 3/2$.

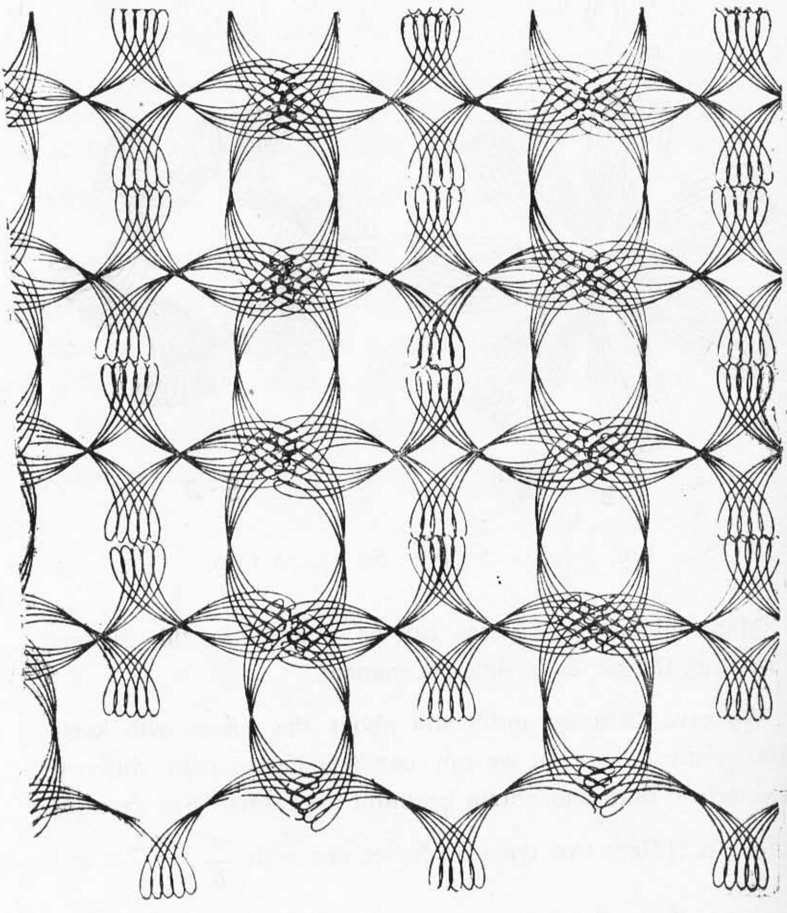


Fig. 6 (a)

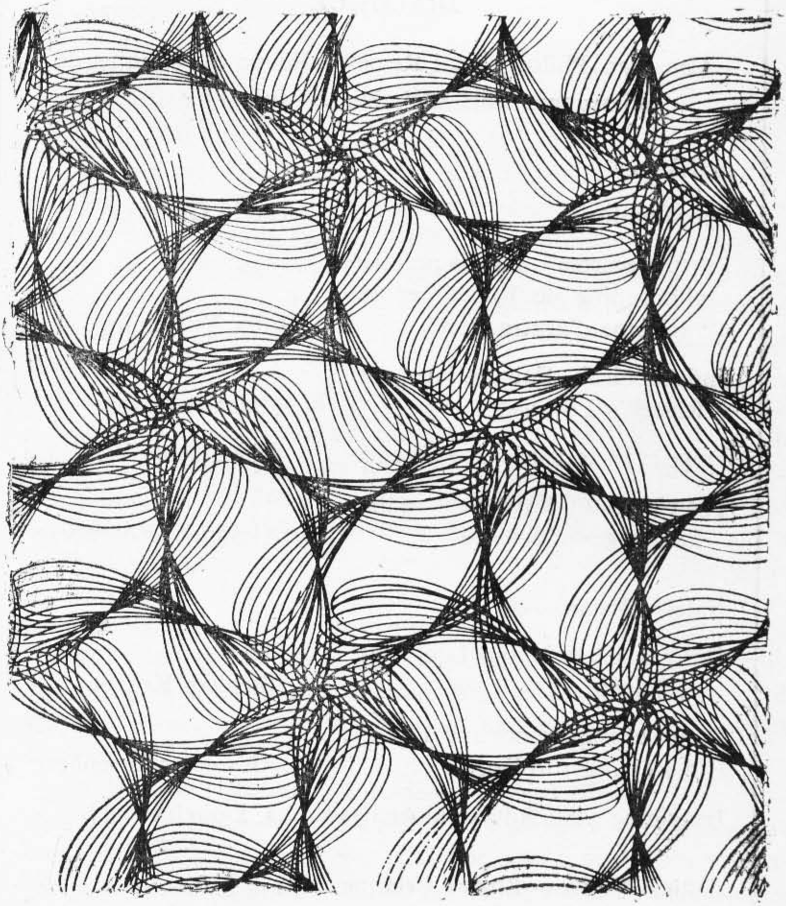


Fig. 6 (b)

Thus we conclude that hypocyclic curves provide us a very big domain for the formation of art-forms.

If the patterns formed by the hypocyclic curves are combined properly, we can get millions of marvellous pieces of Mathematical art and it will not be an exaggeration to say that by using these curves one can begin an entirely new branch of Drawing.

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DIALOGUE

Canadian Philosophical Review—Revue canadienne de philosophie Vol. XIV, No. 2—Juin/June 1975

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