EINSTEIN'S KINEMATICS COMPLETED

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Abstract

Einstein, in his "Zur Elektrodynamik bewegter Körper", gave a physical (operational) meaning to "time" of a remote event in describing "motion" by introducing the concept of "synchronous stationary clocks located at different places". But with regard to "place" in describing motion, he assumed without analysis the concept of a system of co-ordinates.

In the present paper, we propose a way of giving physical (operational) meaning to the concepts of "place" and "co-ordinate system", and show how the observer can determine both the place and time of a remote event. Following Einstein, we consider another system "in uniform motion of translation relatively to the former". Without assuming "the properties of homogeneity which we attribute to space and time", we show that the determination of space and time in the two systems are linearly related. We deduce some novel consequences of our approach regarding faster-than-light observers and particles, "one-way" and "two-way" velocities of light, symmetry, the "group property" of inertial reference frames, length contraction and time dilatation, and the "twin paradox". Finally, we point out a flaw in Einstein's argument in the "Electrodynamical Part" of his paper and show that the Lorentz force formula and Einstein's formula for transformation of field quantities are mutually consistent. We show that for faster-than-light bodies, a simple modification of Planck's formula for mass suffices.

1 Einstein, RADAR and GPS

1.1 Einstein's Synchronous Clocks

In the "Kinematical Part" of Einstein's celebrated "Zur Elektrodynamik bewegter Körper"(all Einstein quotations are from the English translation in [1], except for one quotation from [9]), he remarked: "If we wish to describe the *motion* of a material point, we give the values of its co-ordinates as functions of the time. Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by "time"". He then introduced, "with the help of certain imaginary physical experiments", the concept of "synchronous stationary clocks located at different places", to enable one to determine the "time" of a remote event. This requires, however, setting up an infinitude of clocks located all over "space" and presumably also an infinitude of observers to read these clocks when events occurred at their places. The fact that communication or transmission of "information" from the "remote" observers to a "central" observer takes time has attracted some attention [2, 3, 4, 6, 5, 7].

1.2 Many Clocks or One Clock? The "Radar Approach"

Einstein's approach is different from the so-called "Radar " approach for determining the "distance" of a remote object - and also the "time", although, in practical applications, the distance is more crucial since the time is virtually that of the observation - by a <u>single</u> observer using a <u>single</u> clock and observing an "echo" from the object, of a signal sent by the observer. This "radar" approach

was already envisaged by Nikola Tesla [8] in 1900, and a patent for the use of "Hertzian" waves for detection of a remote object was already granted to Christian Hülsmeyer on April 30, 1904. Perhaps, Einstein was not aware of these developments. Or, if he was, he did not consider them in his study. In fact, Einstein did not use the idea of signaling as a means of determining the time of occurrence of a remote event since he assumed that the network of synchronous clocks had already been set up all over space. (In 1905, transmission and reception of electromagnetic waves was a difficult task. In 2005, it is a commonplace). Of course, setting up a synchronous clock or checking the synchronicity of clocks does require an echo.

In the "radar" approach, by using an echo, a single observer can determine the time and distance of a remote event, and so, there is no need for any transmission of information. Also, if a directional antenna is used the direction towards the remote object (line of sight) can also be determined. This requires setting up three "reference" directions or lines of sight, measuring angles, calculating direction cosines, etc. (In fact, it was this requirement that gave rise to the "three reflecting stations" idea of the present paper.) However, books on the Special Theory of Relativity which mention the radar approach have not spelt out the operational details of determination of place of the event, except in the case of one-dimensional motion in the context of another (moving) observer (for example, the "k-calculus" of Bondi[10]). Also, when the observing radar is mounted on a "moving platform", such as an aircraft, a ship or a land vehicle, the calculations are usually done without making relativistic correction.

1.3 Einstein's "Co-ordinate System" for Space

Einstein also assumed the availability of a "system of coordinates" with "the employment of rigid standards of measurement and the methods of Euclidean geometry". This requires an infinite co-ordinate "grid" with an infinitude of observers all over space to record the position of an event. Of course, the same observers could also read the clocks, thereby determining the space coordinates and time of occurrence of a remote event. In his [9], Einstein says: "If, for instance, a cloud is hovering over Trafalgar Square, then we can determine its position relative to the surface of the earth by erecting a pole perpendicularly on the Square, so that it reaches the cloud. ...We speak of the height of the cloud even when the pole which reaches the cloud has not been erected. By means of optical observations of the cloud from different positions on the ground, and taking into account the properties of the propagation of light, we determine the length of the pole we should have required in order to reach the cloud."

In the present paper, we show how, using only a single clock, one could define a co-ordinate system. It is in this sense that we talk of completing Einstein's kinematics.

1.4 The GPS Approach for Time and Co-ordinate Determination

There is also the method of location used in "Global Positioning System"(GPS)[11]. It is based on the use of spatially separated synchronized clocks located in GPS SVs (Space Vehicles), a la Einstein. But peculiarly enough, in its use, instead of an observer determining the position and time of a remote event, we have an observer (GPS receiver) determining one's own location relative to some specific locations (the master control station and the monitor stations). It is not clear how a GPS could be used by a terrestrial observer to determine the position of a terrestrial or non-terrestrial event. GPS does not involve any echo measurements between SVs and the receiver. Our approach does have some similarity with the radar and GPS approaches, but we use it to define a co-ordinate system and also to study, like Einstein, the relationship between observations by two observers. The GPS method presumes a co-ordinate system but does make correction for relativistic effects (time dilatation and gravity).

1.5 Can we do without an Echo?

If instead of a single observer, we have an observer "assisted" by three "reflecting stations", then by measuring the time-differences between the various signals, direct and indirect, from a remote

object, could an observer determine the place and time of the object? We show below that it is possible to do so, but there are situations where a bivalent ambiguity may arise, i.e., two different determinations are possible for given data. (We give an example of this in Section 3.4.) One would think that an additional reflecting station could be used to resolve the ambiguity, but it turns out that not even finitely many additional reflecting stations would be able to resolve all possible ambiguities. However, if we allow the single observer to obtain an echo from a remote event, then with the help of the three reflecting stations, the place and time of the remote event can be uniquely determined. (An echo involves <u>sending</u> and <u>receiving</u> of a "signal". Note that for "non-luminous" objects, we would still need the echo approach, as is done by bats. Thus a signal sent by the observer to the non-luminous object is reflected back to the observer by the remote object, but it is also reflected by the latter to the three stations which, in turn, reflect it to the observer.) We emphasize that the time recording needs to be done only at one place, thus requiring only one clock, instead of an infinitude of synchronous clocks. The other stations serve merely to reflect the signal sent by the remote object, and so no transmission of information is involved.

1.6   What is an "Observer"?

It would be correct to say that Einstein was the first to take seriously the concept of an "observer". Before his 1905 paper, there was, of course, talk of two "co-ordinate systems" or "reference frames". But even when talking about the "Galilean transformation of co-ordinates" given by $x'(t) = x(t) - vt$, $y'(t) = y(t)$, $z'(t) = z(t)$, $t' = t$, there was no explicit characterization of the two observers involved therein. There was only one omniscient and omnipresent observer, looking at diagrams on paper !

2   Einstein's Two Observers

2.1   What is "Place" ?

In his paper, at the outset, Einstein emphasized that the notion of "time" in describing "motion" was not quite clear and needed a definition. To this, we would like to add that even the notion of "place" is not clear and needs a definition. Since this is a matter of definition, the question as to whether the set of stations, one of which will serve as an observer, are at rest or are moving together does not arise. Of course, we do specify some <u>observable</u> requirements of the set of stations. Following Einstein, we do consider also the possibility of <u>another</u> set of stations, and thus, of defining a different "time and space" for remote events in exactly the same way as the first one. Naturally, the results of the observations by the two systems cannot be related unless one finds <u>out</u> or <u>assumes</u> exactly how the two systems themselves are related. Thus, like Einstein, one may assume that the second system is moving uniformly relative to the first one. In addition, one needs to make some assumptions about the behaviour of the signals themselves. One of these is what Einstein calls a "postulate", namely, that "light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body". His other postulate, which he called the "Principle of Relativity", was that "to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good". Since we are looking at only <u>kinematical</u> considerations, we assume that the second system of observation and calculation be only similar to the first one in its operational aspects and do not consider laws of mechanics, leave alone laws of electrodynamics and optics, except towards the end of the paper (Section 6.9). Einstein assumed the notion of a co-ordinate system for space and so said: "Let us in "stationary" space take two systems of co-ordinates, i.e., two systems, each of three rigid material lines, perpendicular to one another, and issuing from a point" without operationally specifying the meaning of "rigid". When talking about "another system in uniform motion of translation relatively to the former" he said: "Now to the origin of one of the two systems (k) let a constant velocity v be imparted in the direction of the increasing x of the other stationary system (K), and let this velocity be communicated to the axes of the co-

ordinates, the relevant measuring-rod, and the clocks". In our approach, we simply assume that we have one system somehow given or set up, and that we have another system somehow set up which the observer of the first system finds out, on the basis of observations, to be in uniform motion - "relative" to it, of course. (If the other observer is another radar, then that has to have its own set of reference directions. We have not seen any discussion, involving relativistic considerations, of the <u>same</u> object being sighted simultaneously by two radars, not stationary relative to one another.)

2.2   What is an "Event"?

Of course, as Bridgman has remarked, the concept of the "same event" being observed by two systems of observation is not operationally clear, and lurking behind it may be the idea of "absolute time and space". Thus, one talks about a "lightning flash" as an event, but how do the two systems of observers <u>know</u> that they are observing the same event to which they assign perhaps different times and places? Perhaps, a "collision of two particles" or the "onset of a lunar eclipse" are the sort of event about which two observers may agree that they are observing the "same" event.

Einstein tacitly assumes that there must be some definite relations between their findings. Whether or not one should assume the <u>same</u> constant "velocity" of the signal (light) c in the different systems is perhaps a matter of choice, although Einstein <u>deduces</u> "as required by the principle of the constancy of the velocity of light, in combination with the principle of relativity" that "light is also propagated with velocity c when measured in the moving system".

3   Some (philosophical) clarifications:

3.1   <u>Are the considerations here based on some "philosophy" of time?</u>

We assume a minimum, commonsensical, philosophy of time, namely, that human beings have experiences of moment ("at", "when"), duration ("while", "during") and tense (present, past, future), and that they have available some "local" method of observing the time-instant when they have some (momentary) experience. For example, we may assume an "analogue" clock-face or a digital display placed "very close" to the eyes of the observer. Of course, we do not suppose that anything will do under the name of a clock. Certainly, the entire past history and experience of mankind has contributed to the concept of time and development of an artifact called a "clock". Today, we have "atomic clocks" (as in GPS SVs) which are very "stable" and so will remain in synchronism over a very long duration. For those who are so inclined, we could say that a time-instant is represented by an element of the set R of real numbers and that to each momentary experience of an observer, a unique time-instant is assigned. In principle, one could allow a very crude "clock"- think of Galileo counting his pulse beats, or, a child reciting the number-words "one", "two", "three", . . . , when playing a game of hide-and-seek.

3.2   <u>Are the considerations based on some "philosophy" of space ?</u>

Again, we assume a minimum, commonsensical, philosophy of space, namely, that human beings have experiences of place ("at", "here", "there") and distance ("near", "far") through the various senses such as sight, hearing, and touch, and also through their own mobility. We also realize that mankind has developed methods of determination of distance and position, such as by stretching a rope or a cord, using a "rigid" rod, measuring parallax, etc. But in our approach we take the view that the basic observations to be made are only of the time of transmission and reception of "signals" by only one observer. We show that it may be possible to <u>represent</u> these in terms of some spatial concepts, such as 3-dimensional Euclidean geometry, or the set $\mathbb{R}^3$, the set of all triples of real numbers, in a number of different ways known as "co-ordinate systems", but they could also be represented in more abstract ways, for example, by an inner product space. To repeat, we assume that what is observed and recorded is an experience of time; what is defined or determined is a representation in a manner which is certainly influenced by our experience of space. We are certainly not entertaining any conception of a "space-time continuum" or of "spacetime".

Rather than consider time as a fourth dimension of space, we prefer to consider space - as far as our representation is concerned - as three additional dimensions of time! We do have experiences of remote "objects" such as a flying aircraft, the moon, and even galaxies, which cannot be reached by stretching a cord, or by laying out repeatedly a rigid rod, by moving out to the object. Yet we seem to want to extend our concepts of place and distance, based on stretched cords, rigid rods and moving from one place to another, to these remote objects. We show how this can be done.

3.3 Are the considerations based on some "philosophy" of light (as a signal) ?

With Einstein, we assume that light "travels" from one "place" at one "time" to another "place" at another "time" with a constant "velocity" irrespective of what body emits the light and what observer, or system of observations, is used to receive the signal. Of course, in our view, what is observed is only the transmission and reception of the signal at the time shown by the clock. The other times and places of the signal are only inferred, i.e., defined and calculated. Thus, in our proposal, the observer receives four signals, emitted by an object - we are tempted to say, at some place and time - one of them coming directly to the observer from the emitting object, and the other three, after reflection at three stations.

Unlike Einstein, however, we do not assume that velocity of light has been ascertained by experiment to be such and such, because that will involve previous determination of both time and space. Rather, we take velocity as a mere constant that enables us to define distance in terms of time, much like the astronomical way of using light-years. Indeed, we could even take the "velocity" to be unity so that the distance traveled by a light ray is just another name for the duration of its travel. A distance for us is basically associated with the time of travel of light.

3.4 Are the considerations based on some "philosophy" of geometry?

Specifically, are we assuming some "geometry" of "space"? For example, are we assuming the geometry of space to be "Euclidean"? What we are using is a simple kind of "distance" or "metric" geometry [12] wherein we have "points" and "distances between them" - which are non-negative real numbers - satisfying the usual "metric space" axioms, in particular, the "triangle inequality". Some additional concepts may need to be defined, such as "angle" and some properties assumed, such as those involved in the trigonometry of a triangle. So it seems that the "geometry" of an inner product space is adequate. However, as we have emphasized, the choice of a "geometry" for space is only a choice of representation.

First, we need certain extensions to 3-dimensional or "solid" geometry of some Euclidean results in 2-dimensional or "plane" geometry.

Results from 2-dimensional or "plane" geometry: triangle inequalities

Euclid's "Elements" I.20: In any triangle, two sides taken together in any manner are greater than the remaining one. (Hence the term "triangle inequality".)

I.22: Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the three (given) straight lines taken together in any manner should be greater than the remaining one. (It is enough to check that the longest of the three lines is less than the other two lines taken together.)

To construct the triangle, Euclid has to draw or construct some circles. If the construction takes place in a plane, then there are two triangles that satisfy the requirement.

We can show easily that any (non-degenerate) triangle can be co-ordinatized, i.e., represented, non-uniquely in $\mathbb{R}^2$. We say $\mathbb{R}^2$ and not "a plane", because this can be done without assuming that the triangle "lies" in a plane. (If ABC is the triangle, represent A by (0,0), B by (AB, 0) and C by (x, y), with x, y chosen suitably, in two different ways. Of course, this is only one representation and assumes a definition of "distance" in $\mathbb{R}^2$.)

Extension to 3-dimensional or "solid" geometry: tetrahedral inequalities

We need the extension of the above two Euclidean propositions to solid geometry, specifically, for a tetrahedron. Obviously, in any tetrahedron, the three sides of each face of the tetrahedron have to satisfy I.20. But what about the extension of I.22? It seems that there is a partial converse to this. We have not seen this result stated as a theorem in axiomatic treatments of solid geometry.

<u>Partial Converse</u>: Given six straight lines, suppose that some three of them satisfy I.22 and thus can be made into a triangle, say, ABC. If the remaining three straight lines satisfy appropriate inequalities, then a tetrahedron can be erected on triangle ABC such that these remaining straight lines are equal to the three edges of the tetrahedron other than those of the base ABC. Thus, denoting the would-be vertex by D, we have three additional triangles DAB, DBC, DCA, and so with some choice of the three remaining straight lines as the "proposed" edges DA, DB, DC, the appropriate triangle inequalities must be satisfied for the "proposed" faces DAB, DBC, DCA. (To construct the tetrahedron in space, i.e., to "locate" the fourth vertex, one would need to construct appropriate spheres and two different tetrahedra would result.)

Again, we can show easily that the vertices of any (non-degenerate) tetrahedron can be co-ordinatized, i.e., represented non-uniquely in $\mathbb{R}^3$ without visualizing $\mathbb{R}^3$ in terms of three co-ordinate axes . (If DABC is the tetrahedron, represent A by $(0,0,0)$, B by $(AB,0,0)$, C by $(x_1, y_1, 0)$ for suitable non-unique x_1, y_1, and D by (x_2, y_2, z_2) for suitable non-unique x_2, y_2, z_2. Again, this is only one possible representation.)

The following problem of representability can arise in the plane, i.e., in $\mathbb{R}^2$. Suppose ABC is a given (non-degenerate) triangle (i.e., suppose three straight lines satisfying the triangle inequality are given), and we have a representation of it in $\mathbb{R}^2$. Suppose a fourth point D is given, or, rather three more straight lines DA, DB, DC are given, such that the triangle inequalities are satisfied for the triangles DAB, DBC, DCA. Is D representable in $\mathbb{R}^2$? Equivalently, is D coplanar with ABC? The answer is, of course, that D is not necessarily representable in $\mathbb{R}^2$, since a point D can be chosen which is <u>not</u> coplanar with A, B, C. (This is an axiom of "solid" geometry.) If D is representable, its representation is unique.

Now consider an extension of this in solid geometry. Suppose a (non-degenerate) tetrahedron DABC is given and which is, therefore, representable in $\mathbb{R}^3$. Suppose a fifth point E is given and four more straight lines are given which are to be the sides EA, EB, EC, ED. Is the point E co-ordinatizable, i.e., representable in $\mathbb{R}^3$? Obviously, these new straight lines must satisfy the triangle inequalities for the new triangles that are to be formed. But are these inequalities sufficient to guarantee the representability of E as a point of 3-dimensional space? If not, one would have to say that the point E is not representable in $\mathbb{R}^3$, i.e., in 3-dimensional space, and so, one could look for representability in $\mathbb{R}^4$. (To construct a counterexample for representability in $\mathbb{R}^3$, we go to a fourth dimension and choose five appropriate 4-tuples in $\mathbb{R}^4$ such that four of them form a tetrahedron, and so, this tetrahedron can be represented in $\mathbb{R}^3$. But the fifth "point" cannot be represented in $\mathbb{R}^3$. Counterexample 1: choose the 4-tuples as follows. A:$(0,0,0,0)$, B:$(1,0,0,0)$, C:$(0,1,0,0)$, D:$(0,0,1,0)$, E:$(0,0,0,a)$, with $a \neq 0$. The various distances are: AB = AC = AD = 1, AE = a, BC = BD = CD = $\sqrt{2}$, BE = CE = DE = $\sqrt{a^2 + 1}$. The triangle inequalities are satisfied for all the triangles, namely, ABC, ABD, etc. Now, A, B, C, D can obviously be represented in $\mathbb{R}^3$ as $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ respectively, but with this representation we show that E cannot be represented in $\mathbb{R}^3$. We have $BE^2 = AB^2 + AE^2$, so AB is perpendicular to AE in $\mathbb{R}^3$, as are also AC, AD, which is not possible in $\mathbb{R}^3$.) If, however, instead of four straight lines or lengths EA, EB, EC, ED, three differences in lengths, say, EB - EA, EC - EA, ED - EA are specified such that these satisfy appropriate triangle inequalities, then the problem has a (non-unique) solution. (We give an example of non-uniqueness. Counterexample 2: let A be $(0,0,0)$, B:$(1,0,0)$, C:$(0,1,0)$, D:$(0,0,1)$, E:$(-0.1702, -0.1702, -0.1702)$, E′:$(0.0373, 0.0373, 0.0373)$. Then EA = 0.2948, E′A = 0.0646, so EA $\neq$ E′A but EB − EA = E′B − E′A = EC − EA = E′C − E′A = ED − EA = E′D − E′A = 0.9.)

We note that in our approach the co-ordinatization or representation is not any "intrinsic" property of "space" and we are not assuming that space "has" a particular "metric". We are simply choosing a representation which is, a la Poincare (?), convenient (and familiar)!

4 The New Approach:

In the new approach proposed here for the determination of <u>both</u> time and space co-ordinates, we assume a system S consisting of one observer s_0 with a clock and three reflecting "stations" s_1, s_2, s_3. (Einstein used the letter K to denote what he called a "stationary" system and letter k to denote another "moving" system. We will use the corresponding Greek letter Σ to denote the other observation system.) Suppose the observer s_0 observes four time-instants in his clock: one, t_0, of direct reception of a signal emitted by a distant object P when something happens; another, a time instant t_1 of reception of a signal via, i.e., after reflection at, s_1, and similarly, instants t_2, t_3. We assume that these correspond to the "same" event P. We further assume that by using the "echo" method, the observer s_0 ascertains that the three stations are at a constant delay from s_0 and from one another - recall that for us, a distance is a time-difference. Let these one-way time delays between s_0 and s_1, s_2, s_3 be denoted by d_1, d_2, d_3, and the time delay between s_1 and s_2 by d_{12}, etc. We assume that the stations form a non-degenerate tetrahedron, and that the appropriate triangle inequalities are satisfied. (Of course, this can be verified using the distances d_1, etc., and d_{12}, etc.). These correspond to our beliefs that light takes the shortest path between two points, and that the shortest path between two points is a straight line. The observer now postulates that the signal was emitted by the remote object at some (unknown) "time" t, and thus traveled from the object at time t to the observer at time t_0, with $t_0 \geq t$, and thus the "distance" between P and s_0 is $(t_0 - t)$, choosing the velocity of light as "1" i.e., expressing distance in terms of "light-time". Similarly, the signal sent by P at time t must have reached the reflecting station s_1 at time $(t_1 - d_1)$, with $(t_1 - d_1) \geq t$, so that after reflection at s_1, it reached s_0 at time $(t_1 - d_1) + d_1$, i.e., t_1. So the distance between P and s_1 is $(t_1 - d_1 - t)$. Likewise for the signals received from the other stations.

Now, knowing the "distances" d_1, d_2, d_3, and d_{12}, d_{13}, etc., we can co-ordinatize, i.e., represent, these stations s_0, s_1, s_2, s_3 non-uniquely in $\mathbb{R}^3$. In particular we may, and we will, represent s_0 by the origin $(0,0,0)$ of the co-ordinate system. One could also represent them by vectors in a 3-dimensional inner product space, say L, and thus, in particular, s_0 by the zero vector $\mathbf{0}$ of the vector space. We will make use of both these representations. We could denote the representations of s_0, etc., by the same symbols s_0, etc., the context making it clear as to whether we are talking about a station, or the triple in $\mathbb{R}^3$ representing it, or the vector in L representing it.

The problem then would be to determine the representation p of the event P, and the unknown time t, from the measured time instants t_0, t_1, t_2, t_3. That is, we have to determine a triple (x, y, z) or a vector p, and a number t (or, equally well, the number $(t_0 - t)$) such that

$$
\begin{array}{rclcl}
||p - s_0|| &=& t_0 - t &\geq& 0, \\
||p - s_1|| &=& t_1 - d_1 - t &\geq& 0, \\
||p - s_2|| &=& t_2 - d_2 - t &\geq& 0, \\
||p - s_3|| &=& t_3 - d_3 - t &\geq& 0,
\end{array}
\tag{1}
$$

where $|| \cdot ||$ denotes the norm of the triple or vector. Hopefully, this problem has a solution (there are 4 unknowns and 4 equations) and a unique one. Note that $||s_1|| = d_1$, etc., $||s_1 - s_2|| = d_{12}$, etc. (A similar set of equations arises in GPS, except that usually one finds it stated that only three distances are enough. But this is because the GPS receiver is known to be on one particular side of the triangle formed by the three SVs.)

We assume that these time instants t_i satisfy certain additional inequalities, other than the obvious ones given above, which follow from the triangle inequalities for the various triangles formed by P and the four stations. Thus, for example, for the triangle Ps_0s_1 the inequality

$$Ps_0 + s_os_1 \geq Ps_1 \implies (t_0 - t) + d_1 \geq t_1 - d_1 - t,$$

and so

$$t_1 - t_0 \leq 2d_1.$$

We also have, from $Ps_0 + Ps_1 \geq s_0s_1$, the inequality

$$(t_0 - t) + (t_1 - d_1 - t) \geq d_1,$$

so that
$$t_0 + t_1 - 2t - 2d_1 \geq 0.$$

Finally, we have $Ps_1 + s_1s_0 \geq Ps_0$ so that

$$(t_1 - d_1 - t) + d_1 \geq (t_0 - t)$$

and so

$$t_1 \geq t_0.$$

As remarked in Section 3.4 above, this problem may not have a solution - unless we believe that "space" is 3-dimensional. We show that if it has a solution, it will have two solutions, and we will have to choose the one which satisfies the inequalities such as $(t_0 - t) \geq 0$ and others above.

To solve the system, we use the co-ordinate representation of the various points; thus, let P have co-ordinates (x, y, z), and s_i have $(x_i, y_i, z_i)(i = 1, 2, 3)$. Denoting $t_0 - t$ by d, since it is a distance, we have

$$x^2 + y^2 + z^2 = d^2 \tag{2}$$

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = (d + \alpha_i)^2 \tag{3}$$

where $\alpha_i = t_i - d_i - t_0$, and $|\alpha_i| \leq d_i$. Subtracting (3) from (2), we obtain

$$2(x_i x + y_i y + z_i z) - (x_i^2 + y_i^2 + z_i^2) = -2\alpha_i d - \alpha_i^2,$$

and so

$$(x_i x + y_i y + z_i z) = \frac{1}{2}(d_i^2 - \alpha_i^2) - \alpha_i d \quad (i = 1, 2, 3). \tag{4}$$

Since the stations s_0, s_1, s_2, s_3 form a non-degenerate tetrahedron, with s_0 as the origin $(0, 0, 0)$, the determinant corresponding to the left-hand side of the system (4) is non-zero, so (4) has a unique solution for (x, y, z) in terms of the unknown d; suppose it is given by

$$\begin{aligned} x &= a_x d + b_x, \\ y &= a_y d + b_y, \\ z &= a_z d + b_z. \end{aligned} \tag{5}$$

Substituting (5) in (2) gives a quadratic equation for d, namely

$$(a_x^2 + a_y^2 + a_z^2 - 1)d^2 + 2(a_x b_x + a_y b_y + a_z b_z)d + (b_x^2 + b_y^2 + b_z^2) = 0. \tag{6}$$

Now, in principle, the following cases arise. If $(a_x^2 + a_y^2 + a_z^2 - 1) < 0$, then of the two solutions of (6), one is positive and the other negative, and so we obtain a unique positive solution. (Of course, this positive solution may not give x, y, z such that (1) is satisfied. Using counterexample 1 in Section 3.4, with A, B, C, D chosen as s_0, s_1, s_2, s_3, E as P, with $a = 1.5$, $d_1 = d_2 = d_3 = 1$, $d_{12} = d_{23} = d_{31} = \sqrt{2}$, $t_0 - t = 1.5$, $t_i - d_i - t_0 = \sqrt{1 + a^2} + 1$, so that $\alpha_i = t_i - d_i - t_0 = 0.3028$, $a_x = a_y = a_z = -\alpha_i$, so $a_x^2 + a_y^2 + a_z^2 - 1 = -0.7249 < 0$. The positive solution of (6), $d = 0.5164$ gives $x = y = z = 0.2979$ but that does not give $EA = 1.5$.) Could a situation arise in practice where an event cannot be located in $\mathbb{R}^3$, even approximately? In theory, we cannot rule out such a possibility.

If $(a_x^2 + a_y^2 + a_z^2 - 1) = 0$, then there is a unique solution but it may be negative (counterexample 1 with $a = \frac{1}{\sqrt{3}}$.) If $(a_x^2 + a_y^2 + a_z^2 - 1) > 0$, then three cases arise. Equation (6) may have complex solutions in which case (1) has no solutions (counterexample 1 with the following changes: A:$(0, 0, 0, 0)$, B:$(\sqrt{2.61}, 0, 0, 0)$, C:$(0, \sqrt{2.61}, 0, 0)$, D:$(0, 0, \sqrt{0.99}, 0)$, E:$(0, 0, \sqrt{0.99}, 0.1)$, EA = 1, EB = EC = 1.9, ED = 0.1.) If the solutions of (6) are real and both negative (counterexample 1 with $a \leq \frac{1}{\sqrt{3}}$), then again (1) has no solution. Finally, in the third case, (6) may have two unequal positive solutions. (Counterexample 2 in Section 3.4 gives rise to this case.) It is because of this ambiguity, which cannot be resolved in general by using finitely many additional reflecting stations, that we may have to use an echo from the remote object. Thus, the observer would send signal at

time t'_0 to find out that its echo arrives at the same instant t_0 at which the signal possibly generated by the event also arrives and so the time t of the event is immediately determined, "by definition", as Einstein says, to be

$$t = \frac{1}{2} (t'_0 + t_0).$$

In that case, t in (1) is known, and the position p can be uniquely determined from (4) itself, provides solution exists. Henceforth, we will assume that the observer does obtain an echo from the remote object.

We have thus shown that the time of occurrence and place (co-ordinates) of a remote event may be <u>defined</u> and <u>determined</u> by one observer with one clock, with the help of three reflecting stations. These observations involve only transmission of signals by the observer and reception of signals, direct and indirect from the remote event by the observer. We assume that the observer ascertains through various "echo" measurements that the "distances" between the stations remain constant. We next turn to the possibility of envisaging another observer and system of observation.

5   Another Observer:

Like Einstein, we now consider the possibility of another observer, or observation system, exactly like the one in the previous section. This second system Σ will consist of an observer σ_0, with its own clock, and three reflecting stations σ_1, σ_2, σ_3. Σ can also be supposed to assume that the "velocity" of the signal is "1" but this is a matter of choice. We do choose it to be 1. We will assume that, like the first observer, by "echo" experiments, σ_0 can ascertain that the distances between the various stations do not change with (σ_0's) time. Let these distances (and delays) be denoted by δ_1, etc., δ_{12} etc. We also assume that Σ finds $\sigma_0\sigma_1\sigma_2\sigma_3$ to be a non-degenerate tetrahedron so that σ_0 can determine the time of occurrence τ and place (co-ordinates) (ξ, η, ζ) of a remote event P on the basis of observation times τ_0, τ_1, τ_2, τ_3. We will find it advantageous now to use a vector π in place of a triple (ξ, η, ζ), and we will assume that π belongs to a 3-dimensional inner product space Λ. Note that this vector space Λ need not be the same as the vector space L of the first observer, although since both of them are 3-dimensional, they are isomorphic to one another.

5.1   Relation Between the "Times" of the Two Observers

Now, "we" cannot expect to be able to talk about or discover any relation between the determination t, p by the first observer and the determination τ, π by the second observer of the "same" event P, unless "we" assume or discover some relationship between the systems S and Σ themselves. So, with Einstein, we assume that Σ has a "uniform motion of translation relatively to" S. This is something which S (or its observer s_0) can ascertain experimentally, and results in a description of the motion of Σ, i.e., of its stations σ_0, σ_1, σ_2, σ_3 in S's system. So let the motion of these be given by

$$\sigma_0(t) = \overline{\sigma_0} + vt, \quad \sigma_i(t) = \sigma_0(t) + d_i \tag{7}$$

where $\sigma_0(t)$, $\sigma_i(t)$, $\overline{\sigma_0}$, d_i, and v are all vectors, d_i being the position vector of σ_i with respect to σ_0 in S and v the common velocity of the stations σ_0, σ_1, σ_2, σ_3 in S. (The symbols d_i, d_{ij} will no longer denote the distances between the stations s_i of S.)

As noted above, we assume that Σ also finds the distances between the stations σ_0, σ_1, σ_2, σ_3 <u>as measured in that system</u>, to be constant. This is the only assumption we make, namely that, the system σ_0, σ_1, σ_2, σ_3 which is "rigid" in S is also "rigid" in Σ. We show immediately that this assumption has the consequence that the <u>Σ-time</u> at σ_0 must be a constant multiple of the <u>S-time</u> at σ_0. This is a relation which holds for Einstein's formula

$$\tau = \phi(v)\beta(t - vx/c^2)$$

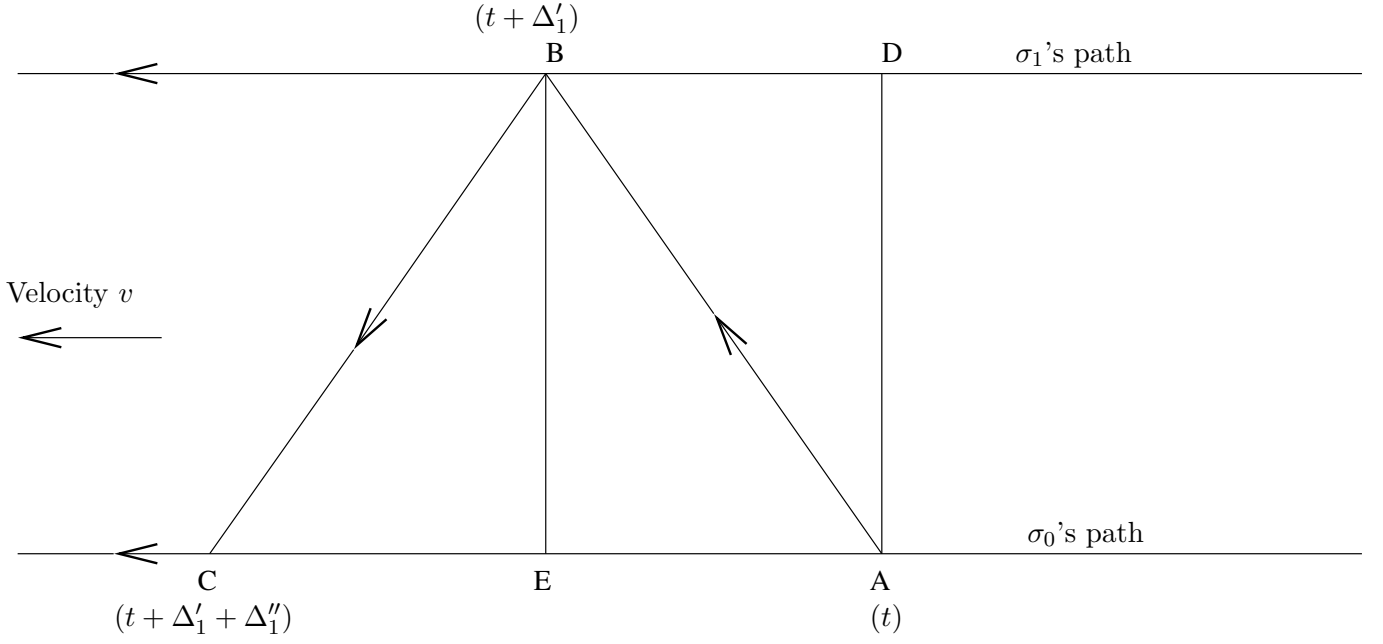with $x = vt$, namely,

$$\tau = \phi(v)\beta(1 - v^2/c^2)t.$$

Figure 1:

(Here, ϕ denotes an undetermined function and $\beta = \frac{1}{\sqrt{1-v^2/c^2}}$.)

Consider Figure 1. (We show a figure only to help "visualize" the derivations; we are <u>not</u> using any "geometry" other than that of an inner product space. Interestingly, there were no figures in Einstein's paper.) It shows the "motion" in S of a signal which starts from σ_0 at A at some time t, reaches σ_1 at B at some later time $(t+\Delta_1')$, and returns to σ_0 at C at a still later time $(t+\Delta_1'+\Delta_1'')$. Here, D is the position of σ_1 at time t, and so $AD = d_1$. E is the position of σ_0 at time $(t+\Delta_1')$, so $EB = d_1$ also, and EB is parallel to AD. We have for the vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{DB}$,

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AD} + \overrightarrow{DB} \\ &= d_1 + \Delta_1' v. \end{aligned}$$

"Squaring" this equation, since $||\overrightarrow{AB}|| = \Delta_1'$, we get:

$$\Delta_1'^{\,2} = d_1^2 + \Delta_1'^{\,2} v^2 + 2(d_1 \cdot v)\Delta_1'$$

where $(d_1 \cdot v)$ denotes the scalar or inner product of the vectors d_1 and v, and so:

$$(1 - v^2)\Delta_1'^{\,2} - 2(d_1 \cdot v)\Delta_1' - d_1^2 = 0. \tag{8}$$

If $v^2 < 1$, the product of the two roots of this quadratic equation is negative; therefore, it has two real roots, one positive and the other negative. We are assuming a "direction" for time, so $\Delta_1' \geq 0$. Thus, it <u>is</u> possible for the signal leaving σ_0 at time t to reach σ_1 at a <u>later</u> time $(t+\Delta_1')$, Δ_1' being the positive root of the quadratic.

Next, for the vectors $\overrightarrow{BC}$, $\overrightarrow{BE}$, $\overrightarrow{EC}$,

$$\begin{aligned} \overrightarrow{BC} &= \overrightarrow{BE} + \overrightarrow{EC} \\ &= -d_1 + \Delta_1'' v. \end{aligned}$$

Squaring this equation and using $||\overrightarrow{BC}|| = \Delta_1''$ we obtain a quadratic for Δ_1'':

$$(1 - v^2)\Delta_1''^{\,2} + 2(d_1 \cdot v)\Delta_1'' - d_1^2 = 0. \tag{9}$$

If $v^2 < 1$, this quadratic, too, has two real roots, one positive and the other negative, so that it <u>is</u> possible for the signal leaving σ_1 at time $t+\Delta_1'$ to reach σ_0 at a <u>later</u> time $(t+\Delta_1'+\Delta_1'')$, Δ_1'' being the positive root of the quadratic.

Thus, if $v^2 < 1$, it is possible for a signal to go from σ_0 to σ_1 and then to return to σ_0, so that σ_0 will be able to "see" σ_1. (In fact, it is easy to see that the roots of the two quadratic equations are negatives of one another, so that the negative root of the first quadratic could have been interpreted as $-\Delta_1''$.) So, the "round-trip" time Δ_1 is given by

$$\Delta_1 = \Delta_1' + \Delta_1'' = \frac{2}{1 - v^2}\ \sqrt{(1 - v^2)d_1^2 + (d_1 \cdot v)^2}.$$

We now assume that σ_0's clock shows a Σ-time τ which is some function ϕ of the S-time t at σ_0:

$$\tau = \phi(t).$$

This is a special case of Einstein's assumption that "to any system of values x, y, z, t, which completely defines the place and time of an event in the stationary system (K), there belongs a system of values ξ, η, ζ, τ, determining that event relatively to the system k".

Now, according to Σ, σ_1 is at a fixed "distance" from σ_0, i.e., the round-trip delay from σ_0 to σ_1 and back to σ_0 is constant. So,

$$\phi(t + \Delta_1' + \Delta_1'') - \phi(t) = \text{constant}.$$

Since this holds for all t, if we assume that ϕ is continuous, then ϕ must be affine, i.e.,

$$\phi(t) = \beta_1 t + \beta_2$$

so that

$$\tau = \beta_1 t + \beta_2$$

at σ_0. We could now follow Einstein, invoke "symmetry" and conclude that $\beta_1 = 1$, rather than Einstein's $\beta = \frac{1}{\sqrt{(1 - v^2/c^2)}}$, because at this point, we do not see any dependence of β_1, β_2 on v. Of course, we could talk about appropriate zero-settings of the two clocks, one at s_0 and the other at σ_0, so that we can take $\beta_2 = 0$. We will see now that β_1 is involved in a relation between the S-distance d_1 between σ_0 and σ_1, and the Σ-distance δ_1 between σ_0 and σ_1. As seen above, the round-trip S-delay between σ_0 and σ_1 is Δ_1 and so the round-trip Σ-delay between σ_0 and σ_1 will be $\beta_1 \Delta_1$, and this must equal twice the Σ-distance δ_1 between σ_0 and σ_1; so $2\delta_1 = \beta_1 \Delta_1$ and thus

$$\delta_1 = \gamma\sqrt{(1 - v^2)d_1^2 + (d_1 \cdot v)^2}.$$

where γ denotes $\frac{\beta_1}{1 - v^2}$.

$$\delta_1^2 \ = \ \gamma^2[(1 - v^2)d_1^2 + (d_1 \cdot v)^2]. \tag{10}$$

What would happen if $v^2 > 1$? The roots of (8) and (9) are real only if $|(d_1 \cdot v)| \geq 2d_1\sqrt{v^2 - 1}$, and under that condition, since their product is positive, the roots are either both positive or both negative. If, further, $(d_1 \cdot v) > 0$, the roots of (8) are both negative. Thus, a signal from σ_0 cannot reach σ_1. On the other hand, if $(d_1 \cdot v) < 0$, then the roots of (9) are both negative. Thus, a signal from σ_1 cannot reach σ_0. Therefore, in either case σ_0 cannot see σ_1! (The case $d_1 \cdot v = 0$ is trivial.) We will comment on the significance of this situation in Section 6.3.

Now we proceed to show that indeed π and τ are related to p and t, and that the relations are linear. We do not assume like Einstein that "the equations must be linear on account of the properties of homogeneity which we attribute to space and time". We have already derived above the linearity of the relation between the S-times and Σ-times at σ_0. We show that τ and t are also linearly related.

As remarked above, we will use a vectorial representation and we emphasize that that the two vector spaces used by S and Σ, namely, L and Λ, need not be identical. We also find it advantageous to work with the "relative" position of p, relative to σ_0 in S, namely

$$\bar{p} = p - \sigma_0(t),$$

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P, t

$\sigma_1, t_1 - \Delta_1''$

replacements

Velocity v

σ_0, t_1

σ_0, t_0

t

t_0'
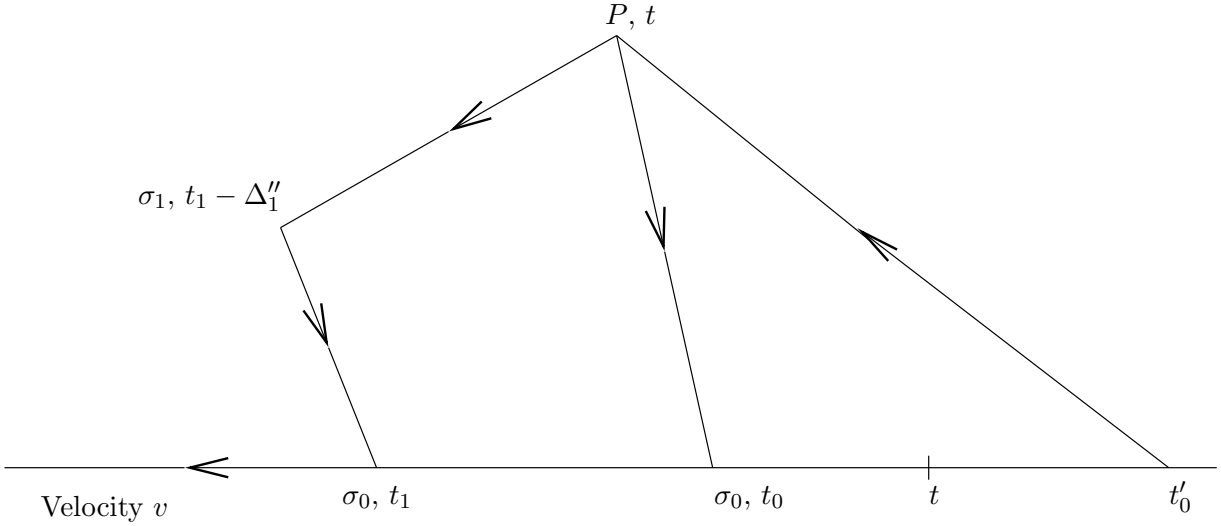
Figure 2:

and thus investigate a possible relationship between $\bar{p}$, t and π, τ.

Firstly, suppose that a signal leaves P with co-ordinate vector p (in S) at time t (in S) and arrives, at some time t_0 (in S), at σ_0, with co-ordinate vector $\sigma_0(t_0)$ (in S). (See Figure 2.) Therefore, we must have

$$||p - \sigma_0(t_0)|| = t_0 - t. \tag{11}$$

But from (7), we have:

$$\sigma_0(t_0) = \sigma_0(t) + v(t_0 - t),$$

so that,

$$||\bar{p} - v(t_0 - t)|| = t_0 - t. \tag{12}$$

Given p and t, or equally well, $\bar{p}$ and t, using (12), we can calculate t_0 as follows. Squaring (12), we get

$$||\bar{p}||^2 - 2(\bar{p} \cdot v)(t_0 - t) + ||v||^2(t_0 - t)^2 = (t_0 - t)^2,$$

and so

$$(t_0 - t)^2 + \frac{2(\bar{p} \cdot v)}{1 - v^2}(t_0 - t) - \frac{||\bar{p}||^2}{1 - v^2} = 0. \tag{13}$$

(Henceforth, when no confusion can arise, we will write "v^2" in place of $||v||^2$, and so on.)

Now, with $v^2 < 1$, this quadratic equation for $(t_0 - t)$ will have two real solutions, one positive and the other negative (because the product of the two roots is negative.) The positive root $(t_0 - t)$ (and so with $t_0 > t$) gives the time of arrival at σ_0 of the signal from P, whereas the negative root $(t_0' - t)$ (and so with $t_0' < t$) gives the time of departure from σ_0 of a signal of which the received signal could well have been an echo from P. (If $v^2 > 1$, then σ_0 will not "see" P. This is similar to the situation with regard to σ_1 discussed above.) The <u>sum</u> of the two roots of the quadratic is

$$-\frac{2(\bar{p} \cdot v)}{1 - v^2},$$

and so we obtain

$$(t_0 - t) + (t_0' - t) = -\frac{2(\bar{p} \cdot v)}{1 - v^2},$$

giving

$$(t_0 + t_0') = 2t - \frac{2(\bar{p} \cdot v)}{1 - v^2}.$$

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But then $\beta_1 t_0$ and $\beta_1 t_0'$ would be the Σ-times of arrival and departure from σ_0 of the signal to P, and so we have immediately the time τ assigned by Σ to P as

$$\begin{aligned}
\tau &= \tfrac{1}{2}\,(\beta_1 t_0 + \beta_1 t_0') \\
&= \beta_1 t - \gamma\,(\bar{p}\cdot v).
\end{aligned} \tag{14}$$

5.2 Relation Between the "Co-ordinates" Determined by the Two Observers

Now, in Σ, the same signal (see Figure 2) leaves from an (unknown) place π at the Σ-time τ that we have calculated, and arrives at σ_0 at S-time t_0, and so, at Σ-time $\beta_1 t_0$, and thus we have

$$\begin{aligned}
||\pi|| &= \beta_1 t_0 - \tau \\
&= \beta_1(t_0 - t) + \gamma(\bar{p}\cdot v).
\end{aligned} \tag{15}$$

From (13), we obtain

$$t_0 - t = \tfrac{1}{2}\left[-2\frac{\bar{p}\cdot v}{1-v^2} + \sqrt{4\frac{(\bar{p}\cdot v)^2}{(1-v^2)^2} + 4\frac{\bar{p}^2}{(1-v^2)}} \,\right]$$

$$= \frac{1}{1-v^2}\left[\sqrt{(\bar{p}\cdot v)^2 + (1-v^2)\bar{p}^2} - \bar{p}\cdot v \,\right]$$

and so we get

$$||\pi|| = \gamma\sqrt{(\bar{p}\cdot v)^2 + (1-v^2)\bar{p}^2}.$$

Squaring both sides, we get

$$\pi^2 = \gamma^2\left[(\bar{p}\cdot v)^2 + (1-v^2)\bar{p}^2 \,\right]. \tag{16}$$

Similarly, suppose the signal, leaving P at t, arrives at σ_0 at S-time t_1 via σ_1. (See Figure 2.) It must have been at σ_1 at S-time $t_1 - \Delta_1''$. So

$$||p - \sigma_1(t_1 - \Delta_1'')|| = t_1 - \Delta_1'' - t.$$

But since

$$\sigma_1(t) = \sigma_0(t) + d_1$$

so

$$p - \sigma_1(t_1 - \Delta_1'') = \bar{p} - d_1 - v(t_1 - \Delta_1'' - t)$$

and thus,

$$||\bar{p} - d_1 - v(t_1 - \Delta_1'' - t)|| = t_1 - \Delta_1'' - t. \tag{17}$$

From (17), we obtain $(t_1 - \Delta_1'' - t)$, just as from (12), we obtained $(t_0 - t)$:

$$t_1 - \Delta_1'' - t = \frac{1}{1 - v^2}\left[\sqrt{\{(\bar{p} - d_1)\cdot v\}^2 + (1-v^2)(\bar{p} - d_1)^2} - (\bar{p} - d_1)\cdot v \,\right].$$

This signal arrives at σ_0 via σ_1 at S-time t_1, and so, at Σ-time $\beta_1 t_1$. (See Figure 3.) So, Σ will assign to the signal the time $(\beta_1 t_1 - \delta_1)$ of arrival at σ_1, and because

$$\delta_1 = \beta_1[\Delta_1'' + \frac{d_1\cdot v}{1 - v^2}],$$

so,

$$\begin{aligned}
||\pi - \delta_1|| &= (\beta_1 t_1 - \delta_1) - \tau \\
&= \beta_1(t_1 - \Delta_1'' - t) + \gamma\{(\bar{p} - d_1)\cdot v\} \\
&= \gamma\left[\sqrt{\{(\bar{p} - d_1)\cdot v\}^2 + (1-v^2)(\bar{p} - d_1)^2} \,\right].
\end{aligned} \tag{18}$$

Squaring both sides of (18), we get

$$\pi^2 - 2\pi\cdot\delta_1 + \delta_1^2 = \gamma^2\left[\{(\bar{p} - d_1)\cdot v\}^2 + (1-v^2)(\bar{p} - d_1)^2 \,\right].$$
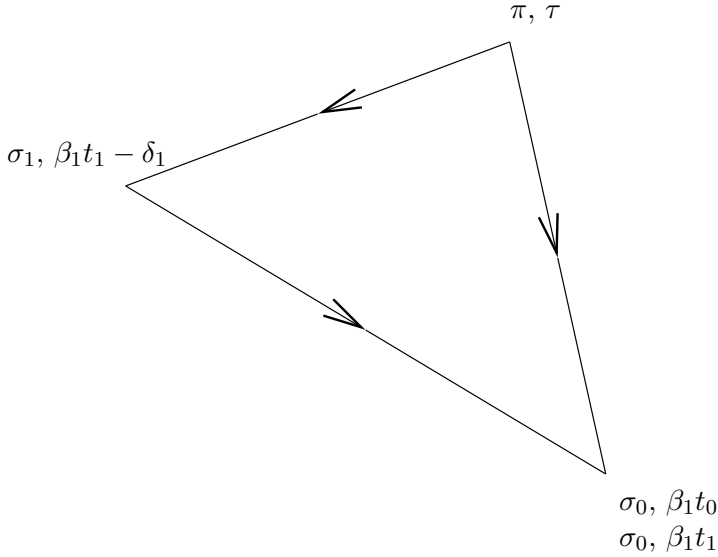
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π, τ

$\sigma_1, \beta_1 t_1 - \delta_1$

$\sigma_0, \beta_1 t_0$
$\sigma_0, \beta_1 t_1$

Figure 3:

Using (16) and (10), this gives

$$\pi \cdot \delta_1 = \gamma^2 \left[(1 - v^2)(\bar{p} \cdot d_1) - (\bar{p} \cdot v)(d_1 \cdot v) \right]. \tag{19}$$

Now, the right-hand side of (19) can be written as $w_1 \cdot \bar{p}$ where

$$w_1 = \gamma^2 \left[(1 - v^2)(d_1) - (d_1 \cdot v)v \right] \tag{20}$$

so that (19) becomes:

$$\pi \cdot \delta_1 = w_1 \cdot \bar{p} . \tag{21}$$

We can similarly obtain two more equations:

$$\pi \cdot \delta_2 = w_2 \cdot \bar{p} , \tag{22}$$

$$\pi \cdot \delta_3 = w_3 \cdot \bar{p} . \tag{23}$$

Since $\sigma_0 \sigma_1 \sigma_2 \sigma_3$ is a non-degenerate tetrahedron in Σ, the vectors δ_1, δ_2, δ_3 are linearly independent, so form a basis for Λ, and so π is a linear combination of δ_1, δ_2, δ_3. This, alongwith the relations (21), (22), (23) implies that there must be a linear transformation T from the vector space L to the vector space Λ:

$$T : L \to \Lambda$$

such that

$$\pi = T(\bar{p}). \tag{24}$$

Thus, we have proved that π and τ are linearly related to $\bar{p}$, the relation being independent of t. The transformation T can be calculated from the equations (21), (22), (23), but knowing that such a transformation exists, we can take a short-cut. Indeed, choose P to be σ_1 itself in S. Then, $\bar{p} = d_1$. But σ_1 is δ_1 in Σ. So we have

$$T(d_1) = \delta_1. \tag{25}$$

Similarly,

$$T(d_2) = \delta_2, \tag{26}$$

$$T(d_3) = \delta_3. \tag{27}$$

(From this, it follows that the vectors d_1, d_2, d_3 form an independent set in L and so, that $\sigma_0 \sigma_1 \sigma_2 \sigma_3$ is a non-degenerate tetrahedron in S also.)
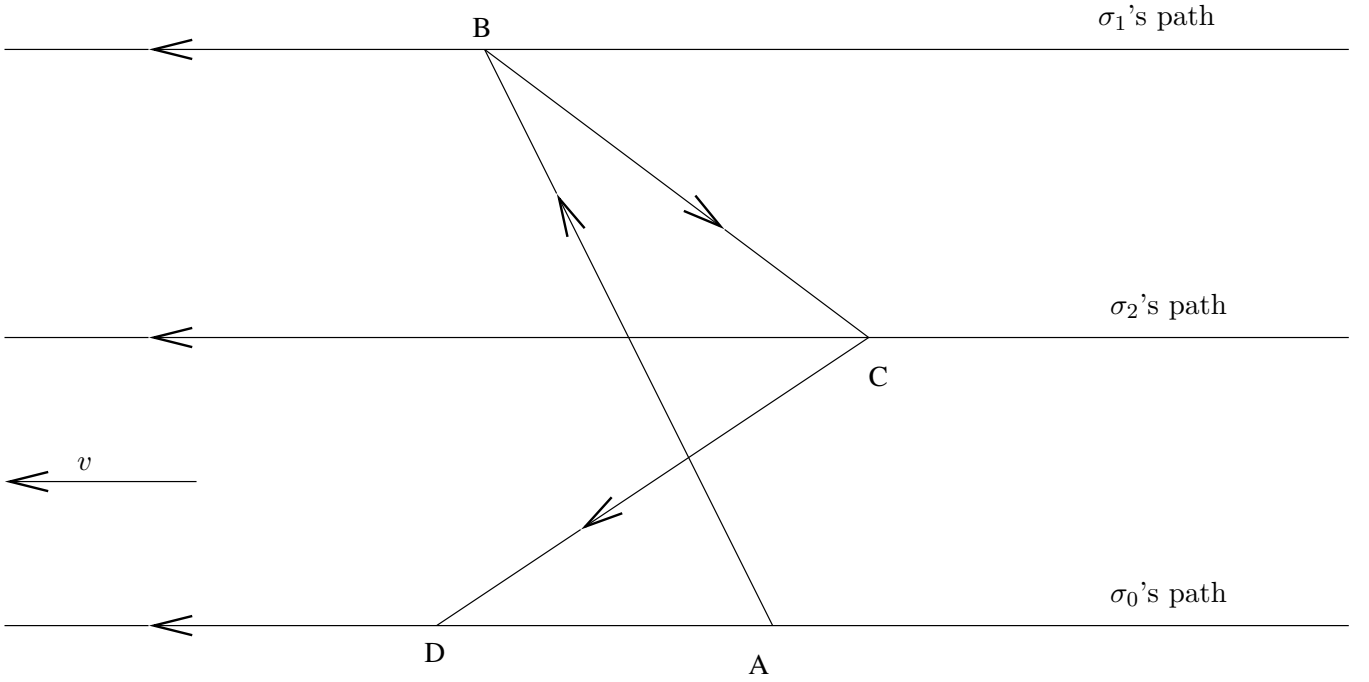
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Figure 4:

Now, the co-ordinatization in Σ involves choosing the representing vectors δ_1, δ_2, δ_3 in Λ. They cannot be chosen arbitrarily, however, because we have the equation

$$||\delta_1|| = \gamma\{\sqrt{(1 - v^2)d_1^2 + (d_1 \cdot v)^2}\}.$$

obtained earlier and so, two more equations for $||\delta_2||$, $||\delta_3||$. Moreover, by considering signals traveling between, say, σ_1 and σ_2 in S and in Σ, we get additional relations for the norms of the vectors δ_{12}, δ_{23} and δ_{31}, these vectors forming the remaining three sides of the tetrahedron $\sigma_0\sigma_1\sigma_2\sigma_3$ in Σ. We can calculate the Σ-distance δ_{12} between σ_1 and σ_2 as follows (see Figure 4).

Consider a round-trip from σ_0 to σ_1 to σ_2 back to σ_0, starting at σ_0 at A at S-time t. This signal will reach σ_1 at B at S-time

$$t + \Delta'_{01}$$

where,

$$\Delta'_{01} = \frac{1}{1 - v^2}\{(d_1 \cdot v) + \sqrt{(1 - v^2)d_1^2 + (d_1 \cdot v)^2}\}.$$

It will reach σ_2 at C at a time Δ'_{12} later where

$$\Delta'_{12} = \frac{1}{1 - v^2}\{(d_{12} \cdot v) + \sqrt{(1 - v^2)d_{12}^2 + (d_{12} \cdot v)^2}\}.$$

Finally, it will return to σ_0 at D at a time Δ''_{20} later where

$$\Delta''_{20} = \frac{1}{1 - v^2}\{-(d_2 \cdot v) + \sqrt{(1 - v^2)d_2^2 + (d_2 \cdot v)^2}\}.$$

Thus, the round-trip time in S is

$$\Delta'_{01} + \Delta'_{12} + \Delta''_{20}$$

and so, in Σ, it is

$$\beta_1(\Delta'_{01} + \Delta'_{12} + \Delta''_{20}).$$

From this, we subtract the Σ-delay δ_1 between σ_0 and σ_1 and the delay δ_2 between σ_2 and σ_0, to obtain the Σ-delay δ_{12} between σ_1 and σ_2:

$$\delta_{12} = \gamma\sqrt{(1 - v^2)d_{12}^2 + (d_{12} \cdot v)^2}$$

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since
$$d_{12} + d_1 - d_2 = 0.$$
Note also that $\delta_{12} = \delta_{21}$.

Thus, the six sides of the tetrahedron $\sigma_0 \sigma_1 \sigma_2 \sigma_3$ in Σ can be calculated from the six sides of the tetrahedron in S and the angles between the vectors $\sigma_0 \sigma_1$, $\sigma_1 \sigma_2$, etc., and the velocity vector v in S. As remarked in Section 3.4, the tetrahedron in Σ can be co-ordinatized or represented by vectors in the inner-product space Λ non-uniquely.

What about description of motion in S and Σ? To be able to define velocity and acceleration, the geometry of an inner-product space suffices. But we do not develop this here.

5.2.1   Theory and Practice of Observers

It is perhaps too "theoretical" or "idealistic" to assume that the distances between the various stations remain constant. To be "practical", one should consider the possibility of these distances (delays, really) varying with s_0's time (or σ_0's time). In such a case, the lengths of the sides of the tetrahedron $s_0 s_1 s_2 s_3$ will change with time. But what do we mean here by "the sides of the tetrahedron at any instant of time"? We had assumed that the observer, with his clock at s_0, finds out by various echo measurements that the delays between the stations remain constant. This allowed us to <u>define</u> the various distances between the stations. Now, if the delays do not remain constant, then we can still give meaning to the distance between, say, s_0 and s_1, at each instant of time on the basis of echo measurement, a la Einstein. Thus, if a signal leaves s_0 at t_1 and returns from s_1 at t_1', the distance between s_0 and s_1 at time $\frac{1}{2}(t_1 + t_1')$ is $\frac{1}{2}(t_1' - t_1)$. Similarly, for the distance between s_0 and s_2, and the distance between s_0 and s_3. But, what does "distance between s_1 and s_2" mean and at what time? The point to be noted is that happenings at the stations s_1, s_2, s_3 cannot be treated as just some events. The stations and observations made on them provide a foundation for <u>defining</u> and calculating the time and position of a remote event. For this, as pointed out earlier, we need <u>all</u> the four stations. The position of a reflecting station, say s_1, cannot be determined using only the three stations s_0, s_2 and s_3. Perhaps, one way out is to adjust the positions of s_1, s_2, s_3 so that the tetrahedron $s_0 s_1 s_2 s_3$ remains unchanged.

Similar considerations will apply to the other observer. It may seem that we could do away with the assumption that the stations σ_0, σ_1, σ_2, σ_3 move together with the same constant velocity if we suppose that system S ascertains their positions continuously. But what about σ_0's own observations of σ_1, σ_2, σ_3? We do not want to assume any a priori relation between S's clock and Σ's clock. We assumed that there was <u>some</u> relation between these clocks, and that what S finds to be "rigid" (the tetrahedron $\sigma_0 \sigma_1 \sigma_2 \sigma_3$) Σ also finds to be rigid. Perhaps, one could assume that Σ borrows S's time, that is to say, that the times at σ_0 are what S assigns to them. It seems that this is exactly what the GPS receiver does; it updates or corrects its clock on the basis of signals received from the space vehicles.

In practice, in GPS, one does expect that the master control station and the monitor stations do remain fixed relative to one another. (Does one actually check this out?) Of course, in the event of an earthquake (or continental drift), these distances could change with time.

6   Some Further Considerations:

6.1   Comparison with Einstein's Formulas: Rectangular Cartesian Co-ordinate Systems

In his derivation, Einstein makes use of the <u>classical</u> "relative position" of an event, relative to a moving observer, namely:
$$x' = x - vt$$
where x is the position of the event in the "stationary system" and v is the velocity of the "moving system". (His "x'" corresponds to our $\bar{p} = p - \sigma_0(t)$.) However, in the final formulas he uses the

variable x. His formulas are:
$$\begin{aligned}
\tau &= \phi(v)\beta(t - vx/c^2), \\
\xi &= \phi(v)\beta(x - vt), \\
\eta &= \phi(v)y, \\
\zeta &= \phi(v)z,
\end{aligned}$$
where $\beta = \frac{1}{\sqrt{1-v^2/c^2}}$.

Using x', rather than x, the formulas become:
$$\begin{aligned}
\tau &= \phi(v)\beta[(1 - v^2/c^2)t - (v/c^2)x'], \\
\xi &= \phi(v)\beta x', \\
\eta &= \phi(v)y, \\
\zeta &= \phi(v)z.
\end{aligned}$$

Using our derivation, we choose (i) the vector spaces L and Λ both to be $\mathbb{R}^3$; (ii) d_1, d_2, d_3, the position vectors of the stations σ_1, σ_2, σ_3, with respect to σ_0 in S to be the three unit vectors i, j, k in $\mathbb{R}^3$, so that S is a rectangular Cartesian co-ordinate system, and (iii) choose the velocity v to be in the direction of the positive x-axis so that $v = ||v||i$. After calculating $||\delta_1||$, $||\delta_2||$, $||\delta_3||$, $||\delta_{12}||$, $||\delta_{23}||$, $||\delta_{31}||$, we see that we can <u>choose</u> the vectors δ_1, δ_2, δ_3 to be

$$\delta_1 = \frac{\beta_1}{1 - v^2}i, \quad \delta_2 = \frac{\beta_1}{\sqrt{1 - v^2}}j, \quad \delta_3 = \frac{\beta_1}{\sqrt{1 - v^2}}k,$$

so that Σ is also a rectangular Cartesian co-ordinate system.

The transformation $T : L \to \Lambda$ is given by the diagonal matrix

$$T = \text{diag}\left[\frac{\beta_1}{1 - v^2}, \frac{\beta_1}{\sqrt{1 - v^2}}, \frac{\beta_1}{\sqrt{1 - v^2}} \right]$$

so that $\pi = Tp$ gives

$$\xi = \frac{\beta_1}{1 - v^2}x', \quad \eta = \frac{\beta_1}{\sqrt{1 - v^2}}y, \quad \zeta = \frac{\beta_1}{\sqrt{1 - v^2}}z.$$

This will agree with Einstein's formulas if we choose

$$\phi(v) = \frac{\beta_1}{\sqrt{1 - v^2}}.$$

Our formula for τ gives

$$\tau = \beta_1\left(t - \frac{vx'}{1 - v^2}\right)$$

which agrees with Einstein's expression with $c = 1$ and $\phi(v) = \frac{\beta_1}{\sqrt{1-v^2}}$.

Hence, we can say that we have generalized Einstein's derivation in two respects: firstly, we have <u>defined</u> the co-ordinate system used operationally, and secondly, we have derived the formulas for any arbitrary direction of the velocity v of the "moving" system relative to the "stationary" system.

6.2 Velocity of Light in the two systems

Where have we used the velocity of light in our derivations? We have used it in defining "distance" in terms of "time difference" and we chose it to be "1" and the same for both systems. It should be clear that the choice of the velocity in each system is arbitrary. We do not imply by this that the velocity of light is arbitrary in practice, or that it cannot be measured in practice. What we mean is that in defining <u>distance</u> in our approach, a constant is used which, by analogy with mechanics of stretched cords, rigid rods and mobile observers, may be called "velocity of light" in our system. It seems to us that the times and distances, and, therefore, also the velocity of light, in the two

systems are non-commensurate in the sense of Kuhn. System S cannot measure velocity of light in its own system, leave alone measure the velocity of light in Σ's system! We have, of course, assumed a "correspondence" between the clocks of S and Σ, but this is something which neither S nor Σ observe. Σ's clock is not observable by S and vice versa. Perhaps, only an omniscient observer can act as a go-between and actually observe that the two times at σ_0 are related, by conducting an appropriate experiment. Thus, one (or someone on one's behalf - unless one travels with the light signal) can send a signal from σ_0 to s_0 at observed Σ-time τ_0, and receive it at s_0 at some observed S-time t from which one can calculate the S-time t_0 of departure of the signal from σ_0, and then find that

$$\tau_0 = \beta_1 t_0,$$

where β_1 is a constant, independent of τ_0.

6.3 "Faster-than-Light" Observer, "Faster-than-light" Particles and Composition of Velocities

"Faster - than - Light" particles have been much discussed in the literature. In our approach, we determine the time and place of an event by using the transmission and reception times of signals. An event is detected or recognized to have happened only if the observer (in S) receives an echo and reflected signals and these signals satisfy the necessary triangle inequalities for a tetrahedron. We have, therefore, no basis for answering the question: "can there be a faster-than-light particle?".

However, if the observer in S does find that the other system Σ is moving faster than light ($||v|| \geq 1$), then as pointed out above, σ_0 of Σ will not be able to see any event P that is seen by S, because no echo from P is possible.

We thus conclude that it is useless to consider a system Σ which moves, relative to S, faster that light because such a system will not see any event seen by S. Note that the reason for this happening is not any expression like "$\sqrt{1 - v^2}$" appearing in our derivation. Einstein had concluded that "for velocities greater than that of light our deliberations become meaningless" for a different reason, namely, the shortening of lengths by the factor $\sqrt{1 - v^2}$.

But the above considerations need not deter us from envisaging faster-than-light particles, because we have shown above that any event P which is seen by S is also visible to Σ, provided $v < 1$. Indeed, we can derive a law of "composition of velocities". Let P move uniformly with a velocity w, i.e., we consider a family of events given by

$$p(t) = wt + \text{constant},$$

and so

$$\bar{p}(t) = (w - v)t + \text{constant},$$

Then, for τ and π, we obtain:
$$\begin{aligned} \tau &= \beta_1 t - \gamma[(w - v) \cdot v] \\ &= \gamma[1 - (w \cdot v)]t \end{aligned}$$

and

$$\pi = T(\bar{p}) = tT(w - v) + \text{constant},$$

so that the velocity of P relative to Σ will be:

$$\frac{T(w - v)}{\gamma[1 - (w \cdot v)]}.$$

This leads to the following possibilities. If $w \cdot v < 1$, which can happen even if $||w|| > 1$, i.e., P travels faster than light, then the "direction" of "time" τ in Σ is same as the "direction" of "time" t in S. But if $w \cdot v > 1$, then there is a time-reversal from S to Σ. However, this time reversal will not be seen by both S and Σ; it is only an omniscient observer who will notice it. If $w \cdot v = 1$, then $\tau = 0$, so that Σ will see P's whole "history" in one moment!

In the special case when w is in the same direction as v, we obtain a simple expression for the magnitude of the relative velocity, which is similar to Einstein's formula. Indeed, if $w = kv$ where k is a real number, then the magnitude of the relative velocity is:

$$|\frac{k-1}{1-kv^2}| \ ||v||.$$

It is interesting to note that Einstein in his paper has <u>nowhere</u> ruled out faster-than-light particles. We show in Section 6.9 that there is no difficulty in considering faster-than-light particles in electrodynamics.

6.4 "One-way" and "Two-way" Velocity of Light

It should be clear that in our approach, velocity of the signal (light) is an "undefined" concept. It is more like a mere number, used to define "distance" in terms of time. So the velocity of light, whether one-way or two-way, is not something which can be measured in our system. Also, "homogeneity" or "isotropy" perhaps are not properties of some independently conceived or experienced "space", but are rather a matter of assumption about the representation of "travel" of light.

6.5 Symmetry

In the calculations above, v was the velocity of Σ as seen by S. What will be the velocity of S as seen by Σ? Will it be $-v$? No, we should not expect it to be $-v$ simply because of the choice involved, of the vector spaces L and Λ, in setting up the co-ordinate systems in S and Σ. But we can use our formula relating π and $\bar{p}$, by taking P to be the origin of S, so that

$$\bar{p}(t) = -\sigma_0(t) = -\bar{\sigma}_0 - vt$$

and since

$$\begin{aligned} \tau &= \beta_1 t - \gamma \ (\bar{p} \cdot v) \\ &= \gamma \ (t + \bar{\sigma}_0 \cdot v) \end{aligned}$$

we have in Σ

$$\pi = T(\bar{p}) = -T(\bar{\sigma}_0) + (\bar{\sigma}_0 \cdot v)T(v) - T(v)\frac{\tau}{\gamma}$$

so that the velocity of S relative to Σ will be

$$-\frac{1}{\gamma}T(v)$$

which need not be $-v$. (v is in the space L, $T(v)$ is in the space Λ.) However, we show below that $||T(v)||$ is equal to $\gamma||v||$, which means that the <u>magnitude</u> of the velocity of S in Σ is the same as the the magnitude of the velocity of Σ in S. In Einstein's special case discussed above, where $v = ||v||i$ and L and Λ are both $\mathbb{R}^3$, $T(v) = -v$, independent of how β_1 is chosen! Indeed, in Einstein's formula, the velocity of K (our S) relative to k (our Σ) is $-v$ independent of how $\phi(v)$ is chosen.(Einstein chooses $\phi(v) = 1$, using some symmetry conditions for the motion.)

To show that $||T(v)|| = \gamma||v||$, we first evaluate the Gram matrix G_δ of the three vectors $\{\delta_1, \delta_2, \delta_3\}$ given by

$$(G_\delta)_{ij} = \delta_i \cdot \delta_j.$$

Using the formulas for δ_i^2 and δ_{ij}^2 and the definitions

$$\delta_{ij} = \delta_j - \delta_i, \quad d_{ij} = d_j - d_i$$

we obtain $\delta_i \cdot \delta_j = (1 - v^2)(d_i \cdot d_j) + (d_i \cdot v)(d_j \cdot v).$

So

$$G_\delta = \gamma^2(1 - v^2)G_d + \gamma^2 \begin{bmatrix} d_1 \cdot v & d_2 \cdot v & d_3 \cdot v \end{bmatrix}^T \begin{bmatrix} d_1 \cdot v & d_2 \cdot v & d_3 \cdot v \end{bmatrix}$$

where G_d is the Gram matrix for the set $\{d_1, d_2, d_3\}$.

Letting $v = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3$, we obtain

$$||v||^2 = \alpha^T G_d \alpha$$

where $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T$.

We then have

$$\begin{aligned} T(v) &= \alpha_1 T(d_1) + \alpha_2 T(d_2) + \alpha_3 T(d_3) \\ &= \alpha_1 \delta_1 + \alpha_2 \delta_2 + \alpha_3 \delta_3 \end{aligned}$$

so $||T(v)||^2 = \alpha^T G_\delta \alpha$.

Now

$$\alpha^T G_\delta \alpha = \gamma^2 (1 - v^2) \alpha^T G_d \alpha + \gamma^2 \alpha^T \begin{bmatrix} d_1 \cdot v & d_2 \cdot v & d_3 \cdot v \end{bmatrix}^T \begin{bmatrix} d_1 \cdot v & d_2 \cdot v & d_3 \cdot v \end{bmatrix} \alpha$$

but since

$$\begin{aligned} \begin{bmatrix} d_1 \cdot v & d_2 \cdot v & d_3 \cdot v \end{bmatrix} \alpha &= \alpha_1 (d_1 \cdot v) + \alpha_2 (d_2 \cdot v) + \alpha_3 (d_3 \cdot v) \\ &= v \cdot (\alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3) \\ &= v^2, \end{aligned}$$

so

$$\begin{aligned} ||T(v)||^2 &= \gamma^2 (1 - v^2) v^2 + \gamma^2 (v^2)^2 \\ &= \gamma^2 v^2, \end{aligned}$$

and thus $||T(v)|| = \gamma ||v||$.

6.6 The "Group Property" and "Inertial Frames of Reference"

We first note that Einstein uses the expression "system of co-ordinates" rather than "inertial frame of reference". We can see from our derivations in Section 5 what role is played by the assumption that Σ is in uniform motion relative to S. If the motion of Σ (even if the tetrahedron $\sigma_0 \sigma_1 \sigma_2 \sigma_3$ stays "rigid") were arbitrary, not much simple could be said about the relation between the times at σ_0 of the two systems, and hence, of the relations between the co-ordinates of any event by them. However, as seen in Section 6.3 above, if we have one more system Σ' in uniform motion at velocity w relative to S, then it will be also in uniform motion relative to Σ (provided $w \cdot v \neq 1$), so that in this sense we have the group property for the set of observers in uniform motion relative to one another. But the relative times and co-ordinates are determined only within an unknown multiplier like β_1. It is convenient, of course, to assume, with Einstein, that

$$\beta_1 = \beta = \sqrt{1 - v^2/c^2},$$

but then only an omniscient observer could verify whether this is so or not.

6.7 Length Contraction and Time Dilatation

We have the relation

$$\delta_1 = \gamma \sqrt{(1 - v^2) d_1^2 + (d_1 \cdot v)^2}$$

and so, the distance between σ_0 and σ_1 in S, namely d_1, need not be the same as the distance between them in Σ, unless β_1 is chosen "properly". But then we have the other two relations also to worry about:

$$\begin{aligned} \delta_2 &= \gamma \sqrt{(1 - v^2) d_2^2 + (d_2 \cdot v)^2} \\ \delta_3 &= \gamma \sqrt{(1 - v^2) d_3^2 + (d_3 \cdot v)^2} \end{aligned}$$

and so we cannot have $\delta_1 = d_1$, $\delta_2 = d_2$, $\delta_3 = d_3$ unless $v = 0$ and $\beta_1 = 1$.

But then what is this contraction or change in "length" in our approach? As we have remarked above, distance in S and distance in Σ are non-commensurate or independent concepts, though their numerical values could be related.

A similar comment could be made with regard to time dilatation. Time (clock) in S and time (clock) in Σ are independent concepts, although, if we assume the "rigidity" condition, they are numerically related; thus, <u>at</u> σ_0

$$\tau = \beta_1 t.$$

But the coefficient β_1 is entirely arbitrary, or rather will be known only to an omniscient observer who can read both the clocks. If we choose β_1 so that $\phi(v) = \frac{\beta_1}{\sqrt{1-v^2}} = 1$, with Einstein, then we do have $\tau = \sqrt{1-v^2}\,t$, a case of time dilatation. With this choice, it turns out that <u>at</u> s_0 in S, there is a time dilatation by the same factor, i.e., $t = \sqrt{1-v^2}\,\tau$.

6.8 The "Twin Paradox"

We observe that neither in Einstein's approach nor in our approach can anything be said about what will happen when an observer, <u>previously</u> at rest, is <u>set</u> in motion. Although Einstein does say that "now to the origin of one of the two systems (k) let a constant velocity v be imparted in the direction of the increasing x of the other stationary system (K)", his derivations <u>nowhere</u> use this conception. It is unfortunate that this way of putting it seems to have led him to formulate what has become known, after Langevin, as the "twin paradox". To emphasize again, Einstein's theory does not say anything as to what happens when a "clock" is set in motion. In our approach, we start with the premise that there are two observers, and that one of them is (already) in motion relative to the other. We, therefore, feel that the speculations by Einstein are not justified. Further, for the clock to return to its starting point, he had to "assume that the result proved for a polygonal line is also valid for a continuously curved line".

6.9 A New Look at the "Electrodynamical Part" of Einstein's Paper

The first part of Einstein's paper, titled "The Kinematical Part", is really about the relation between time and space determination of events in two different observation systems. (In our treatment, the space co-ordinatization is a defined concept.) However, usually, it has been taken to be about "transformation of co-ordinates". This being the case, in electrodynamics, and in particular, as far as Maxwell's equations are concerned, what are the <u>events</u> being studied? The X, Y, Z components of the electric field and the L, M, N components of the magnetic field are not events! In fact, there could be a "vicious cycle" here since the very determination of time and space uses light (signal) which, following Maxwell, is believed to be an electromagnetic phenomenon. Even the application of the "relativistic" approach to mechanics will engender light in the observation of mechanical phenomena. But then this would need a new approach to electrodynamics and the "winning" of Maxwell's equations.

6.9.1 Einstein's "New Manner of Expression" and "Dynamics of the Slowly Accelerated Electron"

Taking Maxwell's equations for granted, as does Einstein, it appears that there is a flaw in the section "Transformation of the Maxwell-Hertz Equations for Empty Space. On the Nature of the Electromotive Forces Occurring in a Magnetic Field During Motion" in the "Electrodynamical Part" of Einstein's paper. To point it out, we need to quote him at length.

"Let the Maxwell-Hertz equations for empty space hold good for the stationary system K, so that we have

$$\frac{1}{c}\frac{\partial \mathrm{X}}{\partial t} = \frac{\partial \mathrm{N}}{\partial y} - \frac{\partial \mathrm{M}}{\partial z}, \quad \frac{1}{c}\frac{\partial \mathrm{L}}{\partial t} = \frac{\partial \mathrm{Y}}{\partial z} - \frac{\partial \mathrm{Z}}{\partial y},$$

$$\frac{1}{c}\frac{\partial \mathrm{Y}}{\partial t} = \frac{\partial \mathrm{L}}{\partial z} - \frac{\partial \mathrm{N}}{\partial x}, \quad \frac{1}{c}\frac{\partial \mathrm{M}}{\partial t} = \frac{\partial \mathrm{Z}}{\partial x} - \frac{\partial \mathrm{X}}{\partial z},$$

$$\frac{1}{c}\frac{\partial \mathrm{Z}}{\partial t} = \frac{\partial \mathrm{M}}{\partial x} - \frac{\partial \mathrm{L}}{\partial y}, \quad \frac{1}{c}\frac{\partial \mathrm{N}}{\partial t} = \frac{\partial \mathrm{X}}{\partial y} - \frac{\partial \mathrm{Y}}{\partial x},$$

where (X, Y, Z) denotes the vector of the electric force, and (L, M, N) that of the magnetic force.

If we apply to these equations the transformation developed in § 3, by referring the electro-magnetic processes to the system of co-ordinates there introduced, moving with the velocity v, we obtain the equations

$$
\begin{aligned}
\tfrac{1}{c}\tfrac{\partial \mathrm{X}}{\partial \tau} &= \tfrac{\partial}{\partial \eta}\left\{\beta\left(\mathrm{N}-\tfrac{v}{c}\mathrm{Y}\right)\right\} &&-\tfrac{\partial}{\partial \zeta}\left\{\beta\left(\mathrm{M}+\tfrac{v}{c}\mathrm{Z}\right)\right\}, \\
\tfrac{1}{c}\tfrac{\partial}{\partial \tau}\left\{\beta\left(\mathrm{Y}-\tfrac{v}{c}\mathrm{N}\right)\right\} &= \tfrac{\partial \mathrm{L}}{\partial \xi} &&-\tfrac{\partial}{\partial \zeta}\left\{\beta\left(\mathrm{N}-\tfrac{v}{c}\mathrm{Y}\right)\right\}, \\
\tfrac{1}{c}\tfrac{\partial}{\partial \tau}\left\{\beta\left(\mathrm{Z}+\tfrac{v}{c}\mathrm{M}\right)\right\} &= \tfrac{\partial}{\partial \xi}\left\{\beta\left(\mathrm{M}+\tfrac{v}{c}\mathrm{Z}\right)\right\} &&-\tfrac{\partial \mathrm{L}}{\partial \eta}, \\
\tfrac{1}{c}\tfrac{\partial \mathrm{L}}{\partial \tau} &= \tfrac{\partial}{\partial \zeta}\left\{\beta\left(\mathrm{Y}-\tfrac{v}{c}\mathrm{N}\right)\right\} &&-\tfrac{\partial}{\partial \eta}\left\{\beta\left(\mathrm{Z}+\tfrac{v}{c}\mathrm{M}\right)\right\}, \\
\tfrac{1}{c}\tfrac{\partial}{\partial \tau}\left\{\beta\left(\mathrm{M}+\tfrac{v}{c}\mathrm{Z}\right)\right\} &= \tfrac{\partial}{\partial \xi}\left\{\beta\left(\mathrm{Z}+\tfrac{v}{c}\mathrm{M}\right)\right\} &&-\tfrac{\partial \mathrm{X}}{\partial \zeta}, \\
\tfrac{1}{c}\tfrac{\partial}{\partial \tau}\left\{\beta\left(\mathrm{N}-\tfrac{v}{c}\mathrm{Y}\right)\right\} &= \tfrac{\partial \mathrm{X}}{\partial \eta} &&-\tfrac{\partial}{\partial \xi}\left\{\beta\left(\mathrm{Y}-\tfrac{v}{c}\mathrm{N}\right)\right\},
\end{aligned}
$$

where

$$\beta = 1/\sqrt{1-v^2/c^2}.$$

Now the principle of relativity requires that if the Maxwell-Hertz equations for empty space hold good in system K, they also hold good in system k; that is to say that the vectors of the electric and the magnetic force—(X′, Y′, Z′) and (L′, M′, N′)—of the moving system k, which are defined by their ponderomotive effects on electric or magnetic masses respectively, satisfy the following equations:—

$$
\begin{aligned}
\tfrac{1}{c}\tfrac{\partial \mathrm{X}'}{\partial \tau} &= \tfrac{\partial \mathrm{N}'}{\partial \eta}-\tfrac{\partial \mathrm{M}'}{\partial \zeta}, & \tfrac{1}{c}\tfrac{\partial \mathrm{L}'}{\partial \tau} &= \tfrac{\partial \mathrm{Y}'}{\partial \zeta}-\tfrac{\partial \mathrm{Z}'}{\partial \eta}, \\
\tfrac{1}{c}\tfrac{\partial \mathrm{Y}'}{\partial \tau} &= \tfrac{\partial \mathrm{L}'}{\partial \zeta}-\tfrac{\partial \mathrm{N}'}{\partial \xi}, & \tfrac{1}{c}\tfrac{\partial \mathrm{M}'}{\partial \tau} &= \tfrac{\partial \mathrm{Z}'}{\partial \xi}-\tfrac{\partial \mathrm{X}'}{\partial \zeta}, \\
\tfrac{1}{c}\tfrac{\partial \mathrm{Z}'}{\partial \tau} &= \tfrac{\partial \mathrm{M}'}{\partial \xi}-\tfrac{\partial \mathrm{L}'}{\partial \eta}, & \tfrac{1}{c}\tfrac{\partial \mathrm{N}'}{\partial \tau} &= \tfrac{\partial \mathrm{X}'}{\partial \eta}-\tfrac{\partial \mathrm{Y}'}{\partial \xi}.
\end{aligned}
$$

Evidently the two systems of equations found for system k must express exactly the same thing, since both systems of equations are equivalent to the Maxwell-Hertz equations for system K. *Since, further, the equations of the two systems agree, with the exception of the symbols for the vectors, it follows that the functions occurring in the systems of equations at corresponding places must agree, with the exception of a factor $\psi(v)$, which is common for all functions of the one system of equations, and is independent of ξ, η, ζ and τ but depends upon v."*

Is the last sentence of the quotation above (our italics) a valid conclusion from the preceding discussion? (Einstein's notation could cause some confusion; he uses the same letter, X, for example, to denote both a function of $(x,\ y,\ z,\ t)$ and of $(\xi,\ \eta,\ \zeta,\ \tau)$, having, of course, the same value at the corresponding quadruples $(x,\ y,\ z,\ t)$ and $(\xi,\ \eta,\ \zeta,\ \tau)$.) Indeed, two pages later ("Theory of Doppler's Principle and of Aberration"), Einstein uses a non-zero solution of the Maxwell-Hertz equations in free space. So, all that Einstein is entitled to say is that the <u>differences</u> X′ − X, Y′ − β(Y − $\tfrac{v}{c}$N), Z′ − β(Z + $\tfrac{v}{c}$M), L′ − L, M′ − β(M + $\tfrac{v}{c}$Z), N′ − β(N − $\tfrac{v}{c}$Y) must satisfy the Maxwell-Hertz equations.

Of course, although it is not <u>necessary</u> that X′ − X = 0, Y′ − β(Y − $\tfrac{v}{c}$N) = 0, etc., it is <u>sufficient</u> in the sense if we <u>define</u> the new functions X′, etc., by equations :

$$
\begin{aligned}
\mathrm{X}' &= \psi(v)\mathrm{X}, & \mathrm{L}' &= \psi(v)\mathrm{L}, \\
\mathrm{Y}' &= \psi(v)\beta\left(\mathrm{Y}-\tfrac{v}{c}\mathrm{N}\right), & \mathrm{M}' &= \psi(v)\beta\left(\mathrm{M}+\tfrac{v}{c}\mathrm{Z}\right), \\
\mathrm{Z}' &= \psi(v)\beta\left(\mathrm{Z}+\tfrac{v}{c}\mathrm{M}\right), & \mathrm{N}' &= \psi(v)\beta\left(\mathrm{N}-\tfrac{v}{c}\mathrm{Y}\right).
\end{aligned}
$$

or, accepting Einstein's argument that $\psi(v) = 1$, by equations which can be legitimately called the "Einstein Field Transformation Equations":

$$
\begin{array}{rclrcl}
X' & = & X, & L' & = & L, \\
Y' & = & \beta\left(Y - \frac{v}{c}N\right), & M' & = & \beta\left(M + \frac{v}{c}Z\right), \\
Z' & = & \beta\left(Z + \frac{v}{c}M\right), & N' & = & \beta\left(N - \frac{v}{c}Y\right).
\end{array}
$$

then these new functions would describe a field in the moving system that would satisfy the Maxwell-Hertz equations as "required" by the Principle of Relativity. Thus, the Principle of Relativity is a guiding principle rather than a physical law.

Einstein goes on to interpret the field transformation equations. We quote again:

"Consequently the first three equations above allow themselves to be clothed in words in the two following ways:—

1. If a unit electric point charge is in motion in an electromagnetic field, there acts upon it, in addition to the electric force, an "electromotive force" which, if we neglect the terms multiplied by the second and higher powers of v/c, is equal to the vector-product of the velocity of the charge and the magnetic force, divided by the velocity of light. (Old manner of expression.)

2. If a unit electric point charge is in motion in an electromagnetic field, the force acting upon it is equal to the electric force which is present at the locality of the charge, and which we ascertain by transformation of the field to a system of co-ordinates at rest relatively to the electrical charge. (New manner of expression.)"

Now, his "old manner of expression" corresponds to the Lorentz force equations. What may have caused Einstein to think of the "new manner of expression"? If $(x(t),\ y(t),\ z(t))$ denotes the instantaneous position of the charge in S, the x-, y- and z- components of the force produced by the field are then given by the Lorentz equations :

$$
\begin{array}{rcl}
F_x & = & X + \dfrac{\dot{y}}{c}N - \dfrac{\dot{z}}{c}M\ , \\[2mm]
F_y & = & Y + \dfrac{\dot{z}}{c}L - \dfrac{\dot{x}}{c}N\ , \\[2mm]
F_z & = & Z + \dfrac{\dot{x}}{c}M - \dfrac{\dot{y}}{c}L\ .
\end{array}
\tag{28}
$$

If we put $\dot{x} = v$, $\dot{y} = 0$, $\dot{z} = 0$, then the expressions on the right-hand-side of the equations above look almost like the expressions on the right-hand-side of the field transformation equations.

His "new manner of expression" has charmed a number of authors because it seems to reduce electrodynamics to electrostatics. But it does not seem to have been realized that the new manner is not useful when the electric charge does not have a uniform motion, or when there is more than one moving charge. Further, we show below that we can hold on to the Lorentz force equations for arbitrary motion of the charge in the stationary system because they will hold in the moving system too provided we make an important change, as suggested by Planck, in the way we handle "mass". Einstein himself seems to have been charmed by his new manner of expression so that he has to consider "the slowly accelerated electron" in the last section of his paper "Dynamics of the Slowly Accelerated Electron", and use language like :

"If the electron is at rest at a given epoch, the motion of the electron ensues in the *next instant of time* [our italics] according to the equations

$$
m\frac{d^2x}{dt^2} = \epsilon X\ ,\quad m\frac{d^2y}{dt^2} = \epsilon Y\ ,\quad m\frac{d^2z}{dt^2} = \epsilon Z
$$

where x, y, z denote the co-ordinates of the electron, and m the mass of the electron, as long as its motion is slow.

Now, secondly, let the velocity of the electron at a given epoch be v. We seek the law of motion of the electron in the *immediately ensuing instants of time*" [our italics].

Einstein applies his field transformation theory to the motion of an electron, by noting that the field quantities X, Y, Z, and X', Y', Z' do determine the force acting on the electron. But what is

"force" acting on a moving body? Einstein finds out that if we "maintain the equation—mass $\times$ acceleration = force", then the electron has two different masses :

$$\text{Longitudinal mass} \quad = \quad \frac{m}{(\sqrt{1 - v^2/c^2})^3} \ ,$$

$$\text{Transverse mass} \quad = \quad \frac{m}{1 - v^2/c^2} \ .$$

Of course, J.J.Thomson and others had deduced earlier that a moving electron has a velocity-dependent mass, but their approach was different from Einstein's. Einstein's approach hinges on his "Theory of Transformation of Co-ordinates and Times", which, in turn, follows from his two Postulates. Further, Einstein suggested: "With a different definition of force and acceleration we should naturally obtain other values for the masses". Perhaps, Planck[13] was inspired by this suggestion.

6.9.2 Lorentz Force and "Variable" Mass

We show below that if <u>accelerative</u> force is defined as suggested by Planck[13], then the equations of motion of a charge in the <u>stationary</u> system under the action of the Lorentz force imply the equations of motion of a charge in the <u>moving</u> system under the action of the Lorentz force. We are also able to see a "reason" why a dynamics in which the mass of a charged body is constant, independent of the velocity, is not compatible with the invariance of the Maxwell-Hertz equations and the Lorentz force equations. It should be pointed out that Planck, in his derivation, surprisingly says that v is to be replaced by $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ in (29) and (30) ("indem überall q an die Stelle von v gesetzt wird", Planck's "q" being $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$).

We will consider here only the special case studied by Einstein where the moving system Σ (Einstein's "k") moves with a constant velocity v in the direction of the x-axis of the stationary system S (Einstein's K). Let us assume that the field quantities X, Y, Z, L, M, N determine the force acting on a unit moving charge as given by <u>Lorentz's formula</u>. If $(x(t), \ y(t), \ z(t))$ denotes the instantaneous position of the charge in S, the x-, y- and z- components of the force produced by the field are then given by (28).

The motion of the charge as seen by the moving observer Σ is given by $(\xi(\tau), \ \eta(\tau), \ \zeta(\tau))$ where

$$
\begin{aligned}
\tau \quad &= \quad \beta(t - vx(t)/c^2), \\
\xi(\tau) \quad &= \quad \beta(x(t) - vt), \\
\eta(\tau) \quad &= \quad y(t), \\
\zeta(\tau) \quad &= \quad z(t),
\end{aligned}
\tag{29}
$$

where $\beta = \frac{1}{\sqrt{1 - v^2/c^2}}$. The velocity components in Σ turn out to be[14] :

$$\frac{d\xi}{d\tau} \ = \ \dot{\xi} \ = \ \frac{\dot{x} - v}{u} \ , \quad \frac{d\eta}{d\tau} \ = \ \dot{\eta} \ = \ \frac{\dot{y}}{\beta u} \ , \quad \frac{d\zeta}{d\tau} \ = \ \dot{\zeta} \ = \ \frac{\dot{z}}{\beta u} \ ,$$

where $u = 1 - \frac{v}{c^2}\dot{x}$, and the accelerative components are given by

$$\ddot{\xi} \ = \ \frac{\ddot{x}}{\beta^3 u^3} \ , \quad \ddot{\eta} \ = \ \frac{u\ddot{y} + \frac{v}{c^2}\dot{y}\ddot{x}}{\beta^2 u^3} \ , \quad \ddot{\zeta} \ = \ \frac{u\ddot{z} + \frac{v}{c^2}\dot{z}\ddot{x}}{\beta^2 u^3} \ .$$

(In his study of the slowly accelerated electron, Einstein effectively sets "at a given epoch" $\dot{x} = v$, $\dot{y} = 0$, $\dot{z} = 0$ but $\ddot{x}, \ddot{y}, \ddot{z}$ may not be zero at that epoch, so that $u = \frac{1}{\beta^2}, \ddot{\xi} = \beta^3 \ddot{x}, \ddot{\eta} = \beta^2 \ddot{y}, \ddot{\zeta} = \beta^2 \ddot{z}$.)

If we now choose for Σ the Lorentz forces given by

$$F_\xi \quad = \quad \mathrm{X}' + \frac{\dot{\eta}}{\mathrm{c}}\mathrm{N}' - \frac{\dot{\zeta}}{\mathrm{c}}\mathrm{M}',$$

$$F_\eta \quad = \quad \mathrm{Y}' + \frac{\dot{\zeta}}{\mathrm{c}}\mathrm{L}' - \frac{\dot{\xi}}{\mathrm{c}}\mathrm{N}',$$

$$F_\zeta \quad = \quad \mathrm{Z}' + \frac{\dot{\xi}}{\mathrm{c}}\mathrm{M}' - \frac{\dot{\eta}}{\mathrm{c}}\mathrm{L}',$$

where X′, Y′, Z′, L′, M′, N′ as functions of (ξ, η, ζ, τ) are related to X, Y, Z, L, M, N by the Einstein relations, then we find (with Planck)[15] that these are related to the Lorentz forces in S as follows:

$$
\begin{aligned}
F_\xi &= F_x - \frac{v}{uc^2}\dot{y}F_y - \frac{v}{uc^2}\dot{z}F_z, \\
F_\eta &= \frac{1}{\beta u}F_y, \\
F_\zeta &= \frac{1}{\beta u}F_z.
\end{aligned}
\tag{30}
$$

Note that these relations <u>do not involve mass</u>. (These relations were derived by Planck[15] exactly as we have done above, and not on the basis of some other principles, as stated by Miller[16].)

We now see that if we assume the mass of a charged body to be a constant, say, m, independent of its velocity, so that the accelerative force components are given by the product (mass × acceleration component), then the equations of motion in S:

$$
m\ddot{x} = F_x \quad, \quad m\ddot{y} = F_y \quad, \quad m\ddot{z} = F_z \quad,
$$

will not imply the equations of motion in Σ:

$$
m\ddot{\xi} = F_\xi \quad, \quad m\ddot{\eta} = F_\eta \quad, \quad m\ddot{\zeta} = F_\zeta \quad.
$$

Suppose that following Planck's suggestion we define the accelerative force to be the time-rate of change of momentum, assuming that the mass has a dependence on velocity given by

$$
m(t) = \frac{m_0}{\left(1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}\right)^{1/2}}
$$

where m_0 is a constant, so that the accelerative force components in S are:

$$
\overline{F}_x = \frac{d}{dt}(m\dot{x}) \quad, \quad \overline{F}_y = \frac{d}{dt}(m\dot{y}) \quad, \quad \overline{F}_z = \frac{d}{dt}(m\dot{z}) \quad.
$$

Assuming that the variation of mass in Σ is given by

$$
\overline{m}(\tau) = \frac{m_0'}{\left(1 - \frac{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2}{c^2}\right)^{1/2}} \quad,
$$

the accelerative force components in Σ are defined by

$$
\overline{F}_\xi = \frac{d}{d\tau}(\overline{m}\dot{\xi}) \quad, \quad \overline{F}_\eta = \frac{d}{d\tau}(\overline{m}\dot{\eta}) \quad, \quad \overline{F}_\zeta = \frac{d}{d\tau}(\overline{m}\dot{\zeta}) \quad.
$$

Fortunately, on using the relations between the velocity and acceleration components in S with those in Σ we find that with this definition of accelerative force, the accelerative forces in Σ are related to the accelerative forces in S by relations, which are analogous to the relations between the Lorentz forces, as follows:

$$
\begin{aligned}
\overline{F}_\xi &= \overline{F}_x - \frac{v}{uc^2}\dot{y}\overline{F}_y - \frac{v}{uc^2}\dot{z}\overline{F}_z \quad, \\
\overline{F}_\eta &= \frac{1}{\beta u}\overline{F}_y \quad, \\
\overline{F}_\zeta &= \frac{1}{\beta u}\overline{F}_z \quad,
\end{aligned}
\tag{31}
$$

if and only if $m_0' = m_0$ (this corresponds to Einstein's conclusion that $\psi(v) = 1$). (We have not seen equations (31) stated <u>explicitly</u> in the literature.)

So, it follows immediately that the equations of motion in S:

$$
\overline{F}_x = F_x \quad, \quad \overline{F}_y = F_y \quad, \quad \overline{F}_z = F_z \quad,
$$

imply the equations of motion in Σ:

$$\overline{F}_\xi \;=\; F_\xi \;\;,\;\; \overline{F}_\eta = F_\eta \;\;,\;\; \overline{F}_\zeta = F_\zeta \;\;.$$

In fact, what we need is that the ratio $\dfrac{\text{mass constant}}{\text{charge}}$ has the same value in S and Σ. If we assume that the charge has the same value in S and Σ, then the constant in the definition of accelerative force has to be the same for both S and Σ, i.e., to be independent of the observer. It could be termed the "absolute mass" or even the "rest mass", since it is the mass when the velocity is zero no matter in which system.

6.9.3 Maxwell-Hertz, Lorentz, Einstein and Planck

Thus, the Einstein transformation of co-ordinates and times, the Einstein transformation of field quantities, the Maxwell-Hertz equations, the Lorentz force equations, and the Planck definition of accelerative force all hang together as well in the stationary system S as in the moving system Σ. Einstein's "new manner of expression" for the force on a moving charge is not required at all and we do not have to agree with Einstein that the "electromotive force plays in the developed theory merely the part of an auxiliary concept". "Newtonian mechanics" can be seen to be valid for charged bodies if instead of Newton's definition of accelerative force, we use Planck's definition. It is interesting that Einstein's insistence that "the laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion" has led to a new definition of accelerative force.

6.9.4 Other Definitions of Accelerative Force

We might ask: are there other definitions of accelerative force which will work? We show immediately that if we assume that the mass is a differentiable function only of $(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, then the only functions that will work are the Planck function and a function that we give below. Denoting $\sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$ by q as before and a desired accelerative force function by $m(q)$, on substituting in (31) and using the fact that $\dot{x}, \dot{y}, \dot{z}$ are arbitrary, in particular, $\dot{x} = v$, we obtain the following differential equation for m:

$$\frac{dm}{dq} = \frac{mq}{(c^2 - q^2)}$$

the solution of which is, not $k_0\{c^2 - q^2\}^{-1/2}$, but

$$k_0|c^2 - q^2|^{-1/2}$$

except for $q = c$. Thus, the solution has two branches:

$$m(q) = k_0(c^2 - q^2)^{-1/2}$$

for $q < c$, and

$$m(q) = k_0(q^2 - c^2)^{-1/2}$$

for $q > c$.

Thus, although the Planck formula for mass presumes that

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 < c^2,$$

i.e., that the body moves more slowly than light, if the body moves faster than light, i.e., if

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 > c^2,$$

we could use the formula :

$$m(t) = \frac{m_0}{\left(\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2} - 1\right)^{1/2}},$$

so that there is no need to agree with Einstein that "Velocities greater than that of light have - as in our previous results - no possibilities of existence" and to entertain any idea of "purely imaginary" mass for faster-than-light bodies. Indeed we find that

$$1 - \left(\frac{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2}{c^2} \right) = \frac{1}{\beta^2 u^2} \left[1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2} \right] ,$$

so that a body moves faster-than-light in S if and only if it moves faster-than-light in Σ.

6.9.5 Charged Bodies Traveling at the Speed of Light

Of course, there is a singularity in the formula for mass when the body moves as fast as light. What should be the mass formula for the case $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = c^2$? (It is possible that this equality may hold over an interval of time, and not at just one time instant.) Note that when $q = c$ we also have $\sqrt{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2} = c$, so that the mass of such a charged body must be the same in both S and Σ, say, m_c. Assuming that the Newtonian definition of accelerative force holds in this case, we find that equations (31) will be satisfied only with $m_c = 0$, i.e., the charged body has zero mass in S and Σ. In that case, the field must be such that the Lorentz force components are zero, i.e., in the equations (28), $F_x = F_y = F_z = 0$. The resulting system of equations for $\dot{x}$, $\dot{y}$, $\dot{z}$ in terms of X, Y, Z, L, M, N has zero determinant. A simple calculation shows that for a solution to exist, the following condition must be satisfied :

$$XL + YM + ZN = 0 ,$$

i.e., the electric field vector must be perpendicular to the magnetic field vector. Under this condition, we can solve the equations for $\dot{y}$, $\dot{z}$ in terms of $\dot{x}$ and the field components, and substitute these in the velocity condition:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = c^2 , \tag{32}$$

obtaining a quadratic equation for $\dot{x}$:

$$(L^2 + M^2 + N^2)\, \dot{x}^2 + 2c(MZ - NY)\, \dot{x} + c^2(Y^2 + Z^2 - L^2) = 0 . \tag{33}$$

This equation has real solutions if and only if

$$L^2 + M^2 + N^2 \geq X^2 + Y^2 + Z^2 ,$$

i.e., the magnetic field is at least as strong as the electric field.

Thus, if the electric and magnetic fields are mutually perpendicular and the magnetic field is at least as strong as the electric field, the question of the possibility of the motion of zero-mass charged bodies at the speed of light amounts to the existence of the solution of equations (28) with their left-hand-sides set equal to zero alongwith the velocity condition (32). Note that the field components are to be evaluated along the motion.

One can immediately verify that in the case of the field corresponding to a plane wave, a motion in straight line in any direction at the speed of light is possible in such a field. Could one conjecture that a photon may be a zero-mass charged body with an "infinitesimally small" charge, just as Einstein thought that "a ponderable material point can be made into an electron (in our sense of the word) by the addition of an electric charge, *no matter how small*"? (With zero accelerative force, the magnitude of the charge has no effect on the motion !) Thus, light, instead of being an electromagnetic wave, could consist of zero-mass charged particles moving in a suitable electromagnetic field. Could some appropriate field allow motion at the speed of light in a circle?

Finally, could there be a "light barrier" so that a motion cannot reach the speed of light even for one instant, whether from "below" or from "above"?

7 Concluding Remarks:

We have extended Einstein's admonition - "a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by "time" "- to apply to the concept of "place" or "co-ordinates", i.e., of a co-ordinate system. We have shown how by considering a system S of an observer with a clock, aided by three reflecting stations, co-ordinates of a remote event can be <u>defined</u> and <u>determined</u>, as also the time of its occurrence.

Considering another system Σ "in uniform motion of translation relatively to" S, we have proved that the co-ordinates and times in Σ are linearly related to the co-ordinates and times in S. The "Lorentz transformation" relating the two could be calculated and turns out to be identical with Einstein's formulas in the special case considered by him.

We have emphasized that the co-ordinates are a matter of <u>representation</u> of the observed data of <u>times</u> of transmission and reception of various signals by the observer. The representation we have used, and the one Einstein implicitly assumed, namely, 3-dimensional Euclidean geometry, is based on our "experience of space", but it is only a representation of the data. This is not to say, of course, that our experiences of seeing a remote object at a certain time in our clock at a certain place are not "real". Indeed, there should be no hesitation in saying that "That train arrives here at 7 o'clock ", or that, "I saw the occultation of Venus by the moon beginning at 7 p.m.".

Is it possible that we may have experiences of departure and arrival times of signals which cannot be represented in 3-dimensional geometry? Could we use some other representation even if the 3-dimensional geometric representation is possible? (We do not mean here alternative co-ordinate systems, such as the spherical-polar, etc.) Also, what we have represented are only certain "points" in the "motion" of light signals ; thus we have supposed that light "leaves" at a certain place at a certain time and "arrives" at another place at another time. Perhaps, we could try to represent or model the entire <u>path</u> of the light signal. This might lead to a different approach to the "General Theory of Relativity".

We have pointed out what appears to be a flaw in the "Electrodynamical Part" of Einstein's paper. We have shown that we can use the Lorentz force formula in the stationary system as well as in the moving system, and that Einstein's interpretation of "electromotive force" as an auxiliary concept is not necessary. Further, faster-than-light motions can be considered without any difficulty.

Lastly, we should perhaps recall Einstein's admonition again with regard to the atomic domain, such as that of an electron, and seek for ourselves operational meanings of time and distance on the atomic scale. Would the same signal suffice for this purpose? What would be a "clock"? What would be "observed" and what would be "defined" and "determined" in terms of what is observed?

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