UNIVERSITY OF PUNE

F.Y.B.Sc. MATHEMATICS Question Bank

Practicals Based on Paper I

First Term: Alegebra

Practical No. 1
Sets and Functions

1. Let $A = \{a, b, c, d\}$. How many elements are there in the power set $\mathcal{P}(A)$? Hence write down $\mathcal{P}(A)$. How many relations are there on the set A?

2. Let $A = \{1, 2, 3, 4\}$. Write down all partitions of A. How many equivalence relations are defined on the set A? Determine the equivalence classes corresponding to each equivalence relation.

3. Let a function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \dfrac{4x-3}{5}$. Show that f is a bijection. Find the formula that defines inverse function f^{-1}.

4. Let the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$. Find the formulae which define the composite functions $f \circ f$, $g \circ g$, $f \circ g$ and $g \circ f$. Is $f \circ g = g \circ f$? Find x for which $f \circ g(x) = g \circ f(x)$.

Practical No. 2
Divisibility in Integers

1. Show that the integers 3927 and 377 are relatively prime. Find the integers m and n such that $31 = m(3927) - n(377)$.
2. Find the values of integers x and y which satisfy $74 = 7469x + 2464y$.
3. Show that $\dfrac{a(a^2+2)}{3}$ is an integer for all integers $a \geq 1$. (by using division algorithm).
4. Find all prime numbers which divide 50!.

Practical No. 3
Congruence Relation on $\mathbf{Z}$

1. Show that $2^5 \equiv -9(\text{mod} 41)$ and hence prove that $41 | 2^{20} - 1$.
2. Find the remainder when $111^{333} + 333^{111}$ is divided by 7.
3. (i) Prepare addition table for $\mathbf{Z}_5$. Write additive inverse of each element in $\mathbf{Z}_5$. (ii) Prepare multiplication table for $\mathbf{Z}_8$. Write multiplicative inverse of the elements of $\mathbf{Z}_8$, which exists.
4. List all integers x with $-10 \leq x \leq 90$, which satisfy $x \equiv 7(\text{mod} 11)$.

Practical No. 4
Complex Numbers

1. Express the following complex numbers in polar form:
   (i) $z = \dfrac{-2}{1+\sqrt{3}i}$ (ii) $z = \dfrac{-1+3i}{2-i}$.

2. Using DeMoivre's theorem, prove the following:
   (i) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
   (ii) $\sin^7\theta = \dfrac{1}{64}\left[35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta\right]$

3. Describe the following regions geometrically:
   (i) $|z-1+i| = 1$. (ii) $0 \leq arg\ z \leq \pi/4$.

4. Find all values of $(-8i)^{1/3}$.

Practical No. 5
Polynomials

1. Find the cubic polynomial $f(x) = a + bx + cx^2 + dx^3$ satisfying $f(0) = 0, f(1) = 1, f(2) = 0, f(3) = 1$.

2. Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$. Given that the roots are in arithmatic progression.

3. (i) Solve the equation $24x^3 - 14x^2 - 63x + 45 = 0$, one root being the double the other.
   (ii) Find the sum of the squares of the roots of the equation $x^3 - 2x^2 + 3x - 4 = 0$.

4. (i) Find the g.c.d. of polynomials $x^3 - 1$ and $x^4 + x^3 + 2x^2 + x + 1$.
   (ii) Consider the equation $x^4 - 5x - 6 = 0$. Find two integral solutions by trial and error method. Also find the other two solutions by using factor theorem.

Practical No. 6
Miscellaneous

1. (i) Give an example of a real valued function f, other than identity function such that
   (a) $f \circ f = identity$ (b) $f \circ f = f$.
   (ii) Find the domain of the following functions:
   (a) $f(x) = \dfrac{x^2 - 3x - 1}{x - 2}$ (b) $f(x) = \sqrt{\sin 2x}$.

2. Define binary operation $*$ on $\mathbf{Z}$ such that $a * b = a + b - ab$. Check whether $*$ is associative. Find the identity element with respect to $*$.

3. Calculate (a) $(-\bar{3})(\bar{4})^{-1}$ in $\mathbf{Z}_7$ (b) $(\bar{5})^{-1} + (\bar{27}) + (\bar{10} - \bar{4})$ in $\mathbf{Z}_{12}$
   (c) $(\bar{12})^2 + \bar{5}(\bar{8}) - \bar{18}$ in $\mathbf{Z}_{19}$.

4. In $\mathbf{Z}_{56}$, find all nonzero pairs $\bar{a}$ and $\bar{b}$, such that $\bar{a} \cdot \bar{b} = \bar{0}$.

5. Calculate (a) $\phi(14) + \phi(18)$ (b) $\phi(22) - \phi(16)$, where ϕ is a Euler's phi-function.

Practicals Based on Paper II

First Term: Calculus

Practical No. 7
Real Numbers

1. Find the solution set of the following inequality

$$2|x| + |x-1| \ < \ 4 \ , x \in \mathbb{R}.$$

2. Find the supremum and infimum of the following sets if exist:

   (a) $S = \{1 - \dfrac{1}{n} \ , \ n \in \mathbb{N}\}$

   (b) $S = \{1 - \dfrac{(-1)^n}{n} \ , \ n \in \mathbb{N}\}$

   (c) $S = \{ \ x^2 + x \ > \ 2, \ x \in \mathbb{R}\}$

3. Let a, b, c, d be real numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$ and one where $bd < ac$.

4. Let $K = \{s + t\sqrt{2}, \ s, t \in \mathbb{Q}\}$. Show that K satisfies the following:

   (a) $x, y \in K$ then $x + y \in K$ and $xy \in K$.

   (b) If $x \neq 0$ and $x \in K$ then $\dfrac{1}{x} \in K$.

Practical No. 8

Sequences

1. By using the definition show that the sequence $\{\frac{2n}{n+1}\}$ converges to 2. Also find N_0 if $\epsilon = 0.1$, 0.01.

2. A sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_{n+1} = \sqrt{3a_n}$. Prove that $\{a_n\}$ is monotonic increasing and bounded. Also find it's limit.

3. Using subsequences show that the sequence $\{\cos n\pi\}$ is not convergent.

4. Check whether the following sequences are Cauchy or not.

   (a) $a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$.

   (b) $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

   Hence test their convergence.

Practical No. 9

Series

1. Discuss the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+3)}{(n+1)^2}$.

2. Discuss the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n(n+1)\sqrt{n+2}}$.

3. Discuss the convergence of $\displaystyle\sum_{n=1}^{\infty} e^{-n^2}$.

4. Discuss the convergence of $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \cdots$.

Practical No. 10
Sequences and Series

1. If $a_n = \sqrt{n+1} - \sqrt{n},\ n \in \mathbb{N}$ then show that $\{a_n\}$ is convergent. Also find it's limit.

2. Find the limits of the following sequences :

   (a) $(1 + \frac{1}{n})^{n+1}$.

   (b) $(1 + \frac{1}{n})^{2n}$.

   (c) $(1 + \frac{1}{n+1})^{n}$.

3. Discuss the convergence of $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots$.

4. By using partial fractions show that $\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha}$ if $\alpha > 0$.

Practical No. 11
Limits

1. Evaluate $\lim_{x\to\infty} \left(\frac{x+6}{x+1}\right)^{x+4}$.

2. Using definition of a limit, prove that $\lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$.

3. Prove that $\lim_{x\to 0} \frac{x^2}{3x+|x|} = 0$.

4. Show that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist but $\lim_{x\to 0} x\sin\frac{1}{x}$ exists.

Practical No. 12
Miscellaneous

1. Show that there exists atleast one irrational number between any two distinct real numbers.

2. Consider the series $1 - 1 + 2 - 2 + 3 - 3 + \cdots$. Let S_n be the sequence of partial sums. Find S_{2n} and S_{2n+1}. Hence show that the series is divergent.

3. If $\lim_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3}$ exists, find the value of a and also evaluate the limit.

4. Evaluate $\lim_{x\to\infty} \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}} + x\tan^{-1}\dfrac{1}{x}\right)$.

5. Let $a_n = \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \cdots + \dfrac{1}{(n+n)^2}$. Show that $\{a_n\}$ is monotone and bounded. Also show that it converges to 0.

Practicals Based on Paper I

Second Term: Analytical Geometry

Practical No. 13

Analytical Geometry of Two Dimensions

1. Under the translation of coordinate axes, the expression $2x^2-3y^2+4y+5$ is transformed into $2x'^2-3y'^2+4x'-8y'+3$. Find the coordinates of new origin.

2. Transform the equation $11x^2+24xy+4y^2-20x-40y-5=0$ when origin shifted to $(2,-1)$ and axes are rotated through an angle $\tan^{-1}(\frac{-4}{3})$.

3. Discuss the nature of the following conic and reduce it into standard (canonical) form. Also find centre, if exists:

   (a) $9x^2+16y^2-54x+64y+1=0$.

   (b) $2x^2-4xy-y^2+20x-2y+17=0$.

4. Discuss the nature of the following conic and find the centre, if exists:$4x^2-12xy+9y^2-52x+26y+81=0$.

Practical No. 14

Analytical Geometry of Three Dimensions

1. Find direction cosines of the straight lines which satisfy the relations $2l+2m-n=0$, $mn+nl+ml=0$.

2. (a) Find the equation of the plane passing (i)through the points $(3,5,1)$, $(2,3,0)$ and $(0,6,0)$(ii) through the point $(2,0,-1)$ and perpendicular to the line whose direction ratios are $3,4,-2$.

(b) Find the equation of the plane through the line $x+y-2z+4=0=3x-y+2z-1$ and parallel to the line with direction ratioes $2,3,-1$.

(c) Find the equation of the plane passsing through the points $(2,3,-4)$ and $(1,-1,3)$ and perpendicular to yz-plane.

3. Find the equation of the perpendicular from the point $(2,4,-1)$ to the line
$\dfrac{x+5}{1}=\dfrac{y+3}{4}=\dfrac{z-6}{-9}$. Find the foot of the perpendicular.

4. Find the point where the line $x+3y-z=6, y-z=4$ meets the plane $2x+2y+z=0$.

Practical No. 15
Sphere

1. Find the equation of the sphere passing through the points $(4,-1,2)$, $(0,-2,3)$, $(1,-5,-1)$ and $(2,0,1)$.

2. Find the length of the chord intercepted on the line
$\dfrac{x+3}{4}=\dfrac{y+4}{3}=\dfrac{z-8}{-5}$ by the sphere
$x^2+y^2+z^2+2x-10y-23=0$.

3. Find the equation of the sphere passing through the circle $x^2+y^2+z^2=9$, $2x+3y+4z=5$ and the point $(1,2,3)$.

4. Find the equations of the spheres that pass through the points $(4,1,0)$, $(2,-3,4)$, $(1,0,0)$ and touch the plane $2x+2y-z=11$.

Practical No. 16
System of Linear Equations(I)

1. Reduce the following matrices to the row echelon form and hence find the rank:
a) $\begin{pmatrix} 2 & 1 & 7 & 3 \\ 1 & 4 & 2 & 1 \\ 3 & 5 & 9 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 & 7 & 3 \\ 1 & 4 & 2 & 1 \\ 3 & 5 & 9 & 2 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & \lambda - 2 & 0 \\ 0 & \lambda - 1 & \lambda + 2 \\ 0 & 0 & 3 \end{pmatrix}$. Find the value of λ for which rank of A is 3.

3. Solve the following system by Guassian elimination method. Find particular solution in each case:

$$\text{(a)}\quad \begin{aligned} \mathrm{x_1 + 2x_2 + x_3 + x_4} &= 0; \\ 3x_1 + 4x_4 &= 2; \\ x_1 - 4x_2 - 2x_3 - 4x_4 &= 2. \end{aligned}$$

$$\text{(b)}\quad \begin{aligned} \mathrm{x - y + 2z - w} &= -1; \\ 2x + y - 2z - 2w &= -2; \\ -x + 2y - 4z + w &= 1; \\ 3x - 3w &= -3. \end{aligned}$$

4. Solve the following system by Guassian elimination method. Find particular solution in each case:

$$\begin{aligned} x - y + 2z - w &= -1; \\ 2x + y - 2z - 2w &= -2; \\ -x + 2y - 4z + w &= 1; \\ 3x - 3w &= -3. \end{aligned}$$

Practical 17
System of Linear Equations (II)

1. Examine the consistency of the following system of equations. If the system is consistent, find the solution.

$$(a)\ \begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 1; \\ 2x_1 - 3x_3 + x_4 &= 2; \\ 4x_1 - x_2 + 2x_3 &= -1; \\ x_2 + x_3 + x_4 &= 1. \end{aligned}$$

$$(b)\ \begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1; \\ x_1 + x_2 - x_3 + x_4 &= 2; \\ x_1 + x_2 - 5x_3 - x_4 &= 3. \end{aligned}$$

2. Examine the consistency of the following system of equations. If the system is consistent, find the solution.

$$\begin{aligned} x_1 - x_2 &= 3; \\ x_2 + x_3 &= 5; \\ 2x_1 + 3x_3 &= 5; \\ 2x_1 - 4x_2 &= 3; \\ x_1 + x_2 + x_3 &= 2. \end{aligned}$$

3. Find the value of the λ such that the following system of equations has a
(i) unique solution (ii) no solution (iii) an infinite number of solutions:
$\lambda x + y + z = 1;$
$x + \lambda y + z = 1;$
$x + y + \lambda z = 1.$

4. Find the value of λ if the following system is consistent:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 5; \\ 3x_1 + 2x_2 - 4x_3 + 7x_4 &= \lambda + 4; \\ x_1 + x_2 - x_3 + 2x_4 &= \lambda - 1. \end{aligned}$$

Practical No. 18
Miscellaneous

1. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplaner. Also find the equation of the plane containing them.

2. Find the distance of point $(2,-1,1)$ from the plane $x+y+z=3$ measured parallel to the line whose direction ratioes are $2,3,-4$.

3. Show that $2x-2y+z+16=0$ is a tangent plane to the sphere
$$x^2+y^2+z^2+2x-4y+2z-3=0$$
and find the point of the contact.

4. Reduce the following matrices to the row echelon form and hence find the rank:
$$\begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & 4 \end{pmatrix}.$$

5. Examine the consistency of the following system of equations. If the system is consistent, find the solution.
$$\begin{aligned} x_1+x_2+x_3+x_4 &= 1; \\ 2x_1-x_2+x_3-2x_4 &= 2; \\ 3x_1+2x_2-x_3-x_4 &= 3. \end{aligned}$$

Practicals Based on Paper II

Second Term: Calculus

Practical No. 19
Continuous Functions - I

1. (a) Give an example of two functions both discontinuous at 0, whose sum is continuous at 0.
   (b) Give an example of two functions both discontinuous at 0, whose product is continuous at 0.
   (c) Do there exist two functions both discontinuous at 0, whose sum as well as product is continuous at 0.

2. Let $f : [0, 1] \to \mathbb{R}$ be defined by
   $f(x)= x$, if x is rational
   $= 1 - x$, if x is irrational.
   Show that f is continuous only at $\frac{1}{2}$.

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by
   $f(x)= 0$, if x is rational
   $= 1$, if x is irrational.
   Show that f is discontinuous at every point of $\mathbb{R}$.

4. The function f is defined on $[0, 3]$ by
   $f(x)= x^2$, if $0 \le x < 1$
   $= 1 + x$, if $1 \le x \le 2$
   $= \frac{6}{x}$, if $2 < x \le 3$.
   Discuss the continuity of f on $[0, 3]$.

Practical No. 20
Continuous Functions - II

1. Let $f : [0,] \to \mathbb{R}$ be defined by
   $f(x)= 0$, if $x = 0$
   $= \frac{x-|x|}{x}$, if $x \neq 0$.
   Discuss the continuity of f on $\mathbb{R}$.

2. Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

3. Prove that there exists a continuous one-one onto function $f : \mathbb{R} \to (-1, 1)$. Find f^{-1}. Is f^{-1} continuous?

4. Find two consecutive integers $n, n+1$ between which a real root of $x^3 + x^2 = 3$ lies.

Practical No. 21
Derivatives and Mean Value Theorems

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|+|x+1|$. Determine whether f is a differntiable function. If so , find the derivative.

2. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(\frac{1}{x^2}$ if $x \neq 0$ and $f(0) = 0$. Show that f is differentiable for all $x \in \mathbb{R}$. Also show that the derivative $f'(x)$ is not bounded on [-1,1].

3. Show that for $0 < a < b$,
$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

4. (a) Find c, of Lagrange's Mean Value Theorem for $f(x) = x^3 - 3x$ on [-1,1].

(b) If $\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$, than show that the equation $a_0x^n + a_1x^{n-1} \cdots + a_n = 0$ has a root in $(0, 1)$.

(c) Using $f(x) = (4 - x) \log x$, show that $x \log x = 4 - x$, for some $x \in (1, 4)$.

(d) Find θ of Cauchy's Mean Value Theorem for $f(x) = \sin x,\ g(x) = \cos x$ in $[0, \frac{\pi}{2}]$.

Practical 22

Successive Differentiation

1. Find n^{th} derivative of the following functions:

   (a) $y = \frac{1}{6x^2+11x+3}$,

   (b) $y = e^x \cos x$,

   (c) $y = x^2 \log x$,

   (d) $y = \tan^{-1}(\frac{2x}{1-x^2})$.

2. By using Leibnitz's theorem prove that if $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, then $(x^2 - 1)y_{n+2} + (2x + 1)xy_{n+1} + (x^2 - m^2)y_m = 0$.

3. If $y = \sin^{-1}(\dfrac{x}{\sqrt{1-x^2}})$ then show that $(1 - x^2)y_{n+2} - (2n + 3)xy_{n+1} - (n + 1)^2y_n = 0$.

4. If $y = \cos(m \cos^{-1} x)$ then show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (m^2 - x^2)y_n = 0$.

Practical 23

Taylor's Theorem and L'Hospital's Rule

1. (a) Use Taylor's series to expand the function $\dfrac{\log(1+x)}{1+x}$ in ascending powers of x up to first four terms.

   (b) Use Taylor's series to expand the function $\log(\sin(x+h))$ in ascending powers of x up to first three terms.

2. Use Maclaurin's series to expand the following functions

   (a) $\log(1+\sin x)$,

   (b) $\sin^{-1} x$,

   (c) $e^{\sin^{-1} x}$.

3. Find the value of $a+b$ such that

$$\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1.$$

4. Evaluate the following limits:

   (a) $\lim\limits_{x\to 0} \ \log(\tan x)^{\tan 2x}$,

   (b) $\lim\limits_{x\to 0} \ (\dfrac{1}{2x^2} - \dfrac{1}{x\tan 2x})$,

   (c) $\lim\limits_{x\to 0} \ (\cos x)^{\dfrac{1}{2x^2}}$.

Practical No. 24
Miscellaneous

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$. Show that
   (a) $f(0) = 0$.
   (b) $f(1) = 1$ or $f(1) = 0$.
   (c) If $f(1) = 0$ then $f \equiv 0$.
   (d) If $f(1) = 1$ then show that f is identity function on $\mathbb{Q}$.
   (e) If $f(1) = 1$ then show that $f(x) > 0$ if $x > 0$. Further, show that f is monotonically increasing function.
   (f) Is f a continuous function?
   (g) Can you determine f?
2. Prove that $3x = 2^x$ for some $x \in (0, 1)$.
3. Use Mean Value Theorem to prove that, for $x > 0$,
$$\frac{x-1}{x} < \ln x < (x-1).$$
4. If $x = \tan(\log y)$ then show that
   $(x^2+1)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$.
5. Find a, if $\lim\limits_{x\to 0} \dfrac{(\sin 2x + a\sin x)}{x^3} = 1$, is finite.